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R. C. Johnson

The Theory and Practice of
Modern Framed Structures

Designed for the Use of Schools and for Engineers in Professional Practice

BY

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IN THREE PARTS

PART I.—Stresses in Simple Structures

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PREFACE

In preparing this edition the authors, after careful consideration and consultation with many who have used the work in the past, have decided to adopt the octavo size and to issue the new edition in parts. It is believed that this arrangement will make the work much more convenient as a text-book, and that it will also be acceptable to those who may use it only as a reference book. From the authors' standpoint it greatly facilitates the work of revision.

It is expected to issue the work in three parts. The present volume, Part I, treats of Simple Structures, including beams and trusses; Part II will treat of structures requiring higher methods of analysis, such as swing bridges, arches, suspension bridges and cantilever bridges, and will include a comprehensive treatment of secondary stresses; Part III will be devoted to the subject of design. Parts I and II correspond, therefore, in a general way with Part I of the old edition, and Part III with Part II.

The present volume covers essentially the same ground as Chapters I to VII and Chapter XV of the old edition. This material has, however, been largely rewritten and considerably expanded. The plan originally adopted of carrying the graphical and algebraic methods along together has been adhered to, with an increased amount of attention given to the graphical processes.

Chapter I, relating mainly to the Historical Development of the Truss, has been left practically as written in 1893 by the lamented Professor Johnson. Chapter II, on the Elements of Analysis, has been considerably enlarged so as to serve more satisfactorily, in connection with Chapter III, as a course in graphic statics. Chapter III has also been much enlarged and includes the complete analysis of the more common types of roof trusses. The revision of Chapters II and III has been almost wholly the work of Mr. W. S. Kinne, Assistant Professor of Structural Engineering in the University of Wisconsin, to whom the

thanks of the authors are due for this work and for many valuable suggestions relative to other parts of the revision.

Chapter IV, dealing with Uniform Loads, remains essentially as rewritten in 1904, with some additions to the introductory matter and to the graphical analysis at various points. Chapter V on Concentrated Loads has been entirely rewritten. The use of influence lines, first brought to the attention of American engineers by Professor Swain in 1887, and introduced in the first edition of this work in 1893, has proven to be a valuable aid in the calculation of stresses, especially for such structures as swing bridges, arches, etc., where the conditions are somewhat complex. While their use in the study of beams or the ordinary single-intersection trusses is of minor importance, it has been thought best to treat the general subject at the outset and to make such use of them in the various classes of problems as seemed expedient. The discussion of equivalent uniform loads is included in this chapter. The use of the influence line in the selection of such loads is explained and a diagram given of a system of equivalent loads for Cooper's E-50 loading for moments for all spans and panel points.

Chapter VI on Lateral Trusses includes new material on Trestles and Towers. In Chapter VII the subject of Deflections and Stresses in Redundant Members has been revised and amplified by the addition of the graphical methods applied to both these problems.

In the new edition the *plus* sign is used for tension and the *minus* sign for compression, this being contrary to former usage. For the usual stress analysis it is quite immaterial which rule is employed, but in problems involving distortions it is more logical and convenient to use the plus sign for tension and the minus sign for compression in order to correspond with the signs of the resulting deformations. This usage appears to be rapidly extending and will doubtless prevail.

Many valuable suggestions have been received from engineers and teachers relative to this revision, and the thanks of the authors are due them for the interest they have taken in this work.

F. E. TURNEAURE.
C. W. BRYAN.

JANUARY, 1910.

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PART I

STRESSES IN SIMPLE STRUCTURES

CHAPTER I

DEFINITIONS AND HISTORICAL DEVELOPMENT

SECTION I.—FORCES AND STRESSES

1. **Forces.**—A *Force* is that which tends to change the state of motion of a body. It becomes evident either as a push or a pull applied at some point of the body.

Forces exerted upon a given body by another body are called *external forces* with respect to the one under consideration. Forces acting in the interior of a body are called *internal forces* or *stresses.* One body may exert a force upon another without contact, as by gravity or magnetic action, or by direct contact producing a direct push or pull.

In the structures to be considered in this work the external forces include all the loads and foundation reactions, including the weight of the structure itself, which act upon and which tend to distort it. These forces are always replaced by their equivalent forces applied at the joints before the direct stresses in the members can be computed.

2. **Strain** is the distortion of a body caused by the application of one or more external forces.* It is measured in units of length, as inches,

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* In popular language "strain" and "stress" are often confused and used indiscriminately. Some authors of repute have also followed the popular usage, but the definitions here given conform to the practice of the leading authorities.

and not in pounds. The proportional, or relative, strain is usually meant, this being the distortion per unit of original length, or in other words, the actual distortion divided by the original length of the member.

3. **Stress** is the resistance of a body to distortion, and can only exist in unconfined bodies when these are solid or plastic. It is measured in pounds or tons the same as the external forces. The stresses resist, or hold in equilibrium, the external forces, but the immediate cause of the stress is the distortion of the body. The external forces upon a framed structure distort the members until the resisting stresses developed in them are sufficient to hold in equilibrium these external or distorting forces. For bodies in equilibrium the external forces and the internal stresses stand in the relation to each other of action and reaction in mechanics. Furthermore, when the stress in one member is resisted by or transmitted to another member or part of a structure it acts upon the latter as an external force.

Sign of the Stress.—In this work tension is called ***plus*** and compression ***minus***. In ordinary stress calculations the opposite convention is quite as convenient, but in problems involving distortions it is more convenient to use plus for tension and minus for compression because the accompanying changes of length are respectively plus and minus.

4. **Relation between Stress and Strain.**—In all solid bodies there is a definite relation between the intensity of the stress and the amount of the accompanying strain. No body is so rigid as to remain unstrained, or undistorted, under the application of any finite external force, however small. Within a certain limit for any particular material a given increase in the external force is always accompanied by a proportionate increase in the strain, or distortion, and this develops a like increase in the stress, or resistance. Thus if any bar of rolled iron or steel be distorted by an external force (pull or thrust) of 28 lbs. per square inch, it will stretch or shorten, as the case may be, an amount equal to one one-millionth part of its length, the internal stress, or resistance to distortion, then coming to be just equal to the external force of 28 lbs. per square inch. An external force of 28,000 lbs. per square inch distorts or strains the bar one one-thousandth part of its length and then develops in the bar a resistance or stress of 28,000 lbs. per square inch. It is evident, however, that this resistance cannot

continue to increase indefinitely in proportion to the distortion. There always comes a time, if the external force continues to increase, when a greater increment of distortion is requisite to develop a given increment of resistance. This point is called

5. The Elastic Limit.—Below this limit the stress and the strain are proportional, equal increments of one always producing equal increments of the other.* Also, below this limit, when the distorting force ceases to act the body returns to its original shape and dimensions and the stress is relieved. If the body be distorted beyond the elastic limit, the strain increases more rapidly than the stress, or than the external force, these two always of necessity being equal to each other, and some of the distortion becomes permanent. That is, when the external force is removed the body does not fully return to its original dimensions, but remains permanently distorted somewhat, or it is said to have "taken a set."

6. The Modulus of Elasticity is the ratio of the stress per unit of area to the relative strain, or distortion, which accompanies it. In other words, it is ***unit stress divided by unit strain,*** or

$$E = \frac{\text{unit stress}}{\text{unit strain}} = \frac{f}{\frac{\alpha}{l}} = \frac{fl}{\alpha}; \quad . \quad . \quad . \quad . \quad (1)$$

where α = distortion or strain (either elongation or compression);
l = original length of part under stress;
f = stress per unit area (pounds per square inch in English units).

Since pounds and inches are the standards used in English, the modulus of elasticity as given and used in all English works must be understood to represent pounds per square inch, the denominator of our fraction in eq. (1) being an abstract number.†

This modulus or ratio is constant within the elastic limit. Beyond that it steadily decreases until it reduces to zero in the case of solid metals where the material becomes plastic and draws out or compresses under a constant load. All working stresses are, or should be, well

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* This is known as Hooke's Law and was originally expressed by the Latin phrase "*Ut tensio sic vis.*"

† If this denominator could become unity, which it never can in solids, then the fraction, or E, would represent the number of pounds per square inch required to stretch a body to twice its original length, and the modulus of elasticity is sometimes so defined.

within the elastic limit, and hence for all such stresses this ratio is constant for any given material. When it is known, the resisting stresses can be found for a known distortion, or the distortion may be computed for a known stress. The determination of this ratio requires very delicate measuring apparatus with the most careful and expert handling. Tabular values given for these moduli for different materials in standard works are not very reliable. Thus, for all the rolled irons and steels this modulus is remarkably constant, being perhaps always between 26,000,000 and 31,000,000 lbs. per square inch for the ordinary temperatures, while the tensile strength of these metals will vary from 45,000 lbs. per square inch for wrought-iron and soft steel to over 200,000 lbs. per square inch for hard-drawn steel wire. It is an extremely valuable property of engineering materials, and is used to great advantage by the scientific designer.

EXAMPLES.—The following examples are given to illustrate some of the uses to be made of the modulus of elasticity. In solving these problems, take the modulus of wrought-iron as 27,000,000, and of steel as 29,000,000; of cast-iron as 12,000,000; and of timber as 1,500,000 lbs. per square inch.

1. A steel-wire cable 5 miles long and one square inch in solid section is pulled with an average force of 15,000 lbs. What is the strain, or stretch?

2. The rim of a cast-iron fly wheel 10 feet in diameter is subjected to a tensile stress of 5,000 lbs. per square inch from the centrifugal force. How much is its diameter increased?

3. If an iron or steel rail 30 feet long is prevented from expanding, what will be the stress in it per square inch resulting from a rise of temperature of 80° F., the coefficient of expansion being taken at 0.0000065?

4. A series of wooden posts superposed upon each other in a building to a total height of 60 feet are subjected to an average compressive stress of 1,000 lbs. per square inch. How much will be the settlement at the top from this cause?

SECTION II.—THE TRUSS AND ITS ELEMENTS

7. **A Truss** is a framed or jointed structure designed to act as a beam while each member is usually subjected to longitudinal stress only, either tension or compression.

8. **The Struts** are those members which are compressed endwise, and which therefore have developed in them compressive resistances or stresses. Struts are sometimes called Posts, or Columns.

9. **The Ties** are those members which are extended, and which thus have developed in them tensile resistances or stresses.

10. **The Upper and Lower Chords** are composed of the upper and lower longitudinal members respectively. When the loads are downward and the truss is supported at its ends, the upper chord is always in compression and the lower chord always in tension. The spaces between the chord joints are called *panels.*

11. **The Web Members** are those which join the two chords. They are alternately in tension and compression, or the struts and ties alternate in the web system.

12. **A Counterbrace** is a member which is designed to resist both tensile and compressive strains. That is, for one position of the load the member may be elongated, while for another it may be compressed, and hence at different times it must resist both extension and compression. When two or more external forces act upon it, some of which tend to compress the member and others to extend it, it is evident that only the algebraic sum of these forces really acts upon the member. It is subjected to a stress, therefore, equal to and of the same kind as the algebraic sum of all the external forces acting upon it.

Hence, also, a tension member or tie may resist a compressive external force without becoming a counterbrace, so long as this compressive force is smaller than another extending force which also acts. Similarly with struts, they can resist tensile external forces without becoming ties, so long as these are less than other compressive forces which continue active. The residual stress is tension in the one case and compression in the other, for which only the member is designed, and therefore we may say that both struts and ties may resist the contrary external forces without becoming counterbraces, the stresses in the member always being of one sign or kind.

13. **Mains and Counters.**—A main member, whether strut or tie, is one which acts when the entire structure is loaded. A counter is one which acts only for particular partial loads.

14. **Illustration.**—In Fig. 1, which represents a Pratt truss, the compression members, or struts, are shown by double lines, and the tension members, or ties, by single lines. When the end posts are inclined as in the figure, they would seem to belong about as much to the upper chord as to the web system. They are usually spoken of separately as the "inclined end-posts" or the "batter-braces."

The members $e F$, $f G$, $F g$, and $G h$ are counters, or counter-ties.

There are no counterbraces in this truss; that is, no members have to resist both tension and compression.

The tension members $B c$, $C d$, $D e$, $E f$, $g H$, $h I$, $i J$, and $j K$ are main tie-rods. The members $B b$ and $K k$ are not elements of the truss proper, since they serve only to carry the loads at b and at k to the hip-joints.

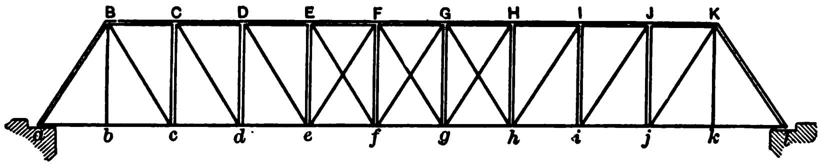


FIG. 1.

15. The Action of a Truss.—Since a truss is a jointed structure composed of rigid but elastic members so arranged as to form an unyielding combination, it must be composed of an assemblage of rigid polygons. But the only rigid polygonal figure is a triangle. A truss must therefore be composed of an assemblage of triangles. Any assemblage of triangles fastened together at their apices, consecutive figures having sides in common, is a truss, and will act as a beam. In Fig. 1 the triangles are all right-angled. A load placed at joint e, for instance, is carried by the truss as a beam to the abutments at a and l. The part of this load which goes to the left abutment may be conceived as being carried up to D, down to d, up to C, down to c, up to B, and then down down to a, where it passes to the ground. The part which goes to the right passes over the path, e, F, f, G, g, H, h, I, i, J, j, K and l. Thus for such a load the ties $e F$ and $f G$ are put under stress, while $E f$, $F g$, and $G h$ are idle, as well as the post $E e$ and the hangers $B b$ and $K k$. If all these idle members were removed, the truss would stand under this particular loading, since it would still remain an assemblage of triangles properly joined. If the load were placed f, $E f$ and $f G$ would be under stress, while $e F$ and $F g$ would be idle. If the two middle joints f and g were loaded equally, the part of the load at f going to the right is just balanced by the part of the load at g going to the left, and hence there is no stress in the intermediate web members $f G$, $F g$, $F f$, $g H$. The counter-ties $e F$ and $G h$ are also idle.

In Fig. 2 we have generalized conceptions of a truss. They are assemblages of triangles, adjacent figures having a common side, and exemplify the generic idea of a truss.

A truss is not weakened from its want of symmetry or from its sagging in the middle, provided all the members are properly proportioned to carry their loads.

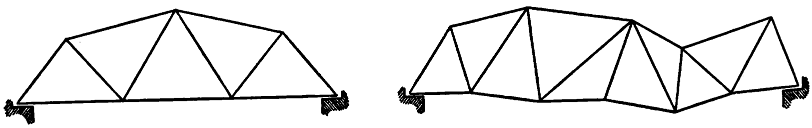


FIG. 2.

A Through-bridge is one in which the roadway is carried directly at the bottom-chord joints, with lateral bracing overhead between the top-chord joints, thus inclosing a space through which the load passes.

A Deck-bridge is one in which the roadway is carried directly at the top-chord joints, or on the upper chords themselves. The trusses are usually placed closer together than on through-bridges, the roadway extending over them.

A Pony Truss is a low truss of short span, with the roadway carried at the bottom joints, but not of sufficient height to allow of the upper lateral bracing. The trusses are stayed, or held in place, by bracing, connected with the floor system.

SECTION III.—HISTORICAL DEVELOPMENT OF THE TRUSS IDEA

16. Primitive Systems.—The earliest forms of truss were built of timber, the progressive development of the forms used for bridge purposes being shown in the following figures.

Figs. 3*a*, 3*b*, 3*c* are three forms of truss construction employed by Palladio, a famous Italian architect, 1560–80. These trusses were built entirely of timber, and are believed to be the earliest examples of a scientific use of the truss element, the rigid triangle. Palladio wrote an elaborate illustrated treatise on architecture in which these and other forms of truss have been preserved.

The mastery of the principles of truss construction did not follow

the practice of Palladio, and so fine an example of the use of the truss element is not found again for nearly three hundred years.

Fig. 4 represents one span, 170 feet long, of a bridge over the Rhine at Schaffhausen, built in 1758 by Ulric Grubenmann, a carpenter by trade but really a great engineer. He afterwards built a wooden bridge

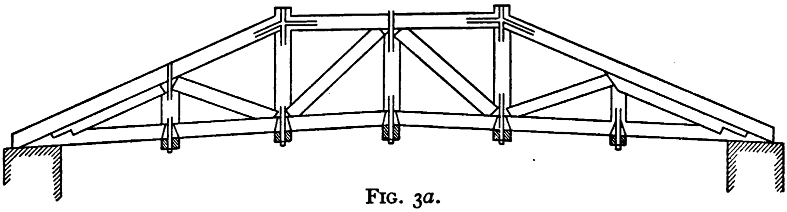


FIG. 3*a*.

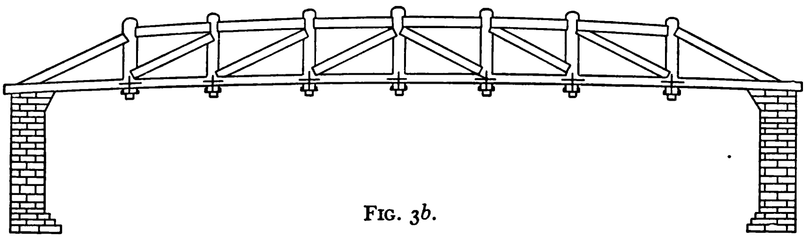


FIG. 3*b*.

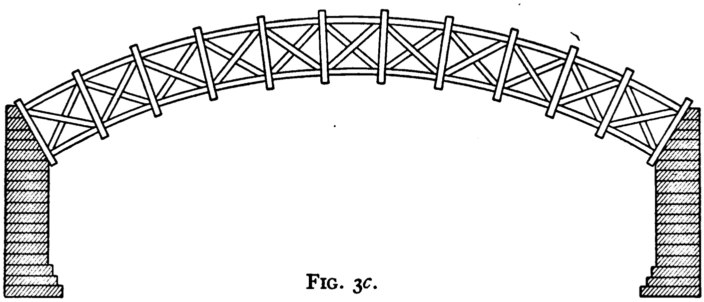


FIG. 3*c*.

of similar design, 366 feet long, near Baden. Both these bridges served their purpose till destroyed by Napoleon in 1799.

Fig. 5 represents the "Permanent Bridge" over the Schuylkill River at Philadelphia, built by Mr. Timothy Palmer of Newburyport, Mass., in 1804.* The middle span was 195 feet and the side spans

* See Paper on *American R. R. Bridges*, by Theodore Cooper, *Trans. Am. Soc. Civ. Engrs.*, Vol. XXI, p. 1.

were 150 feet each. It was covered in and continued in service till 1850, when it was replaced by a bridge for railroad purposes.

In 1804 Mr. Theodore Burr built a bridge over the Hudson River at Waterford, in four spans of 154, 161, 176, and 180 feet clear span, respectively, after the pattern shown in Fig. 6. All members were of timber, and counter-struts were inserted the entire length, thus giving

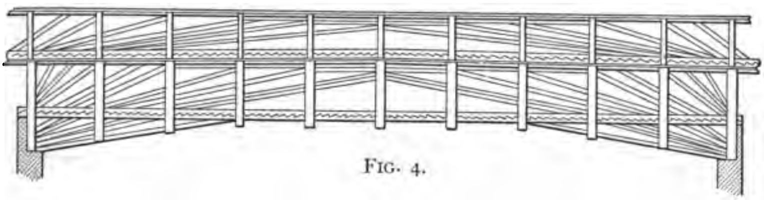


FIG. 4.

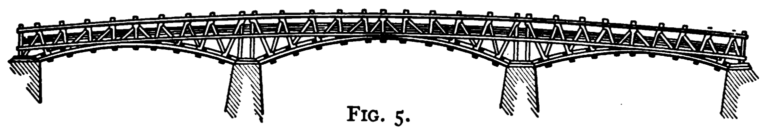


FIG. 5.

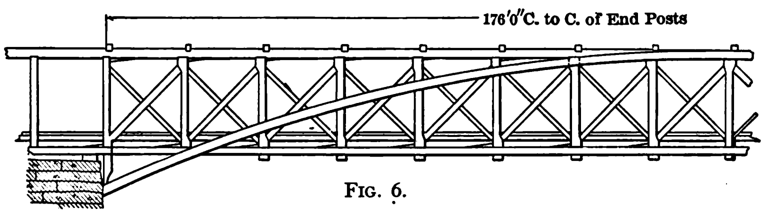


FIG. 6.

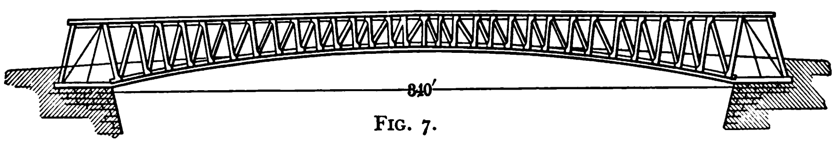


FIG. 7.

great rigidity. This is probably the most scientific design for an all-wooden bridge ever invented, and for a half-century it stood unrivalled for cheapness and efficiency for highway purposes in this country. These bridges were always covered in, the covering extending many feet beyond the end of the truss proper.

In Fig. 7 is shown a view of the Colossus Bridge over the Schuylkill River at Philadelphia, built in 1812 by Mr. Lewis Wernwag. It was 340 feet clear span, and marked a great advance on previous practice

in America in the length of span. It was destroyed by fire in 1838. These three gentlemen, Palmer, Burr, and Wernwag, were, up to 1840, the leading bridge engineers of America.

All the above types of bridges, Figs. 4–7, are composite forms and not simple trusses. The last three are really trussed arches, the arch carrying the greater part of the load, while the truss prevented undue distortions in it. The Burr truss would have had considerable strength alone if the bottom chord had been more efficiently spliced.

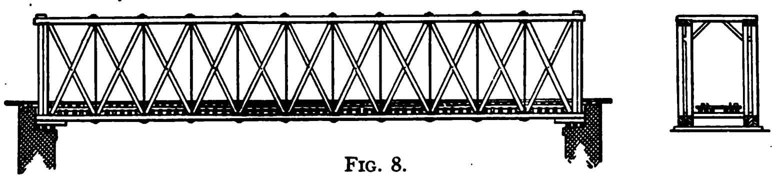


FIG. 8.

17. **Early Forms of Simple Trusses.**—It is believed that the Howe truss, Fig. 8, is the earliest form of simple truss for long spans ever built for bridge purposes, except those of Palladio.* It was patented in the United States in 1840 by William Howe. Although but a modification of the Burr truss, it was designed to carry the entire load as a truss, without any aid from arches or from auxiliary struts projecting out from the abutments. It is constructed of timber except the vertical tie-rods, which are of iron. It has been repeated thousands of times in this

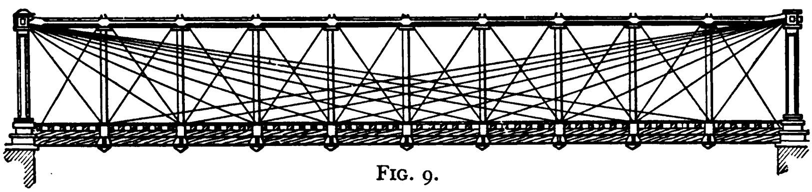


FIG. 9.

country on both railroads and highways, and will long continue a favorite form of bridge for new lines of railroad, and for wagon-roads in places where timber is cheap and the site distant from a railway station. It will be fully treated in the second part of this work.

The earliest type of iron truss construction in the United States was the invention of Mr. Wendell Bollman, Fig. 9. It is really a trussed

* Long's truss was braced out from the abutments and hence did not give simple vertical reactions.

beam, each joint-load being carried directly to the ends of the upper chord by two inclined tension members. There is no stress in the lower chord. It is therefore a kind of multiple suspension system and not properly a truss at all. It was largely used from 1840 to 1850 on the Baltimore and Ohio Railroad.

An improvement on the Bollman truss, as shown in Figs. 10 and 10*a*, was made about 1851 by Mr. Albert Fink, then assistant engineer

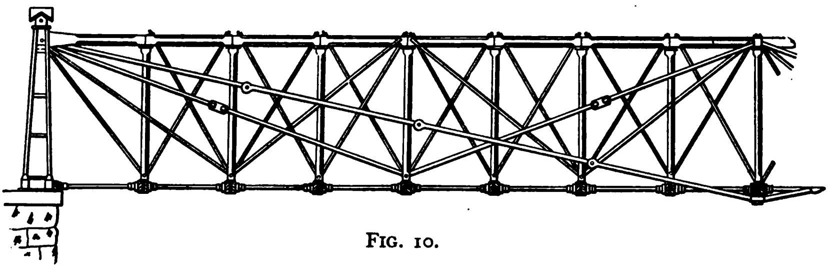


FIG. 10.

on the B. & O. R. R. These are called "Fink trusses," the loads being carried at the lower and upper joints respectively. This design has been more often used for deck-bridges, though many through-spans have been built on this plan.

In both the Bollman and the Fink trusses the compression members were usually made of cast-iron, and in neither case was there any stress

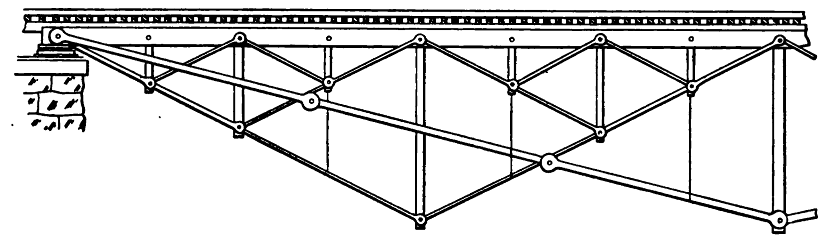


FIG. 10 *a*.

in the lower chord for static loads. Bollman trusses were very soon abandoned, but Fink trusses were built for some twenty-five years, or down to about 1876, the compression members being made finally of wrought-iron. Pins and eye-bars were freely but not exclusively used for joints in the Bollman and Fink trusses.

The first near approach to the modern style of iron truss-bridges

was made by Squire Whipple* in 1852, in a bridge of 146 feet span, which he built 7 miles north of Troy, N. Y., on the Rensselaer and Saratoga Railway (Fig. 11). The ties in the web system extend over two panels, and it is therefore called a "double-intersection" truss. The lower chord was composed of wrought-iron links passing over wrought-iron trunnions in the bottoms of the posts. All the compression members were of cast-iron, and it was pin-connected in both upper and lower chords. This form of arrangement of members is still known as the Whipple truss.

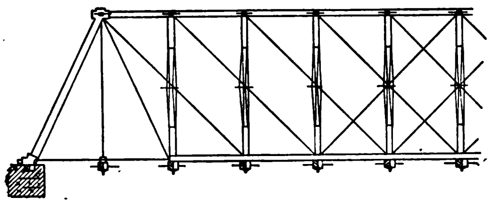


FIG. 11.

In 1863 Mr. John W. Murphy first used wrought-iron for all the compression members in truss construction, but still used cast-iron in joint-blocks and pedestals. On account of this improvement by Mr. Murphy in the Whipple truss, the modern wrought-iron or steel double-

* Mr. Squire Whipple, C.E., a philosophical instrument-maker of Utica, N. Y., seems to have the distinguished honor of being the first man who ever correctly and adequately analyzed the stresses in a truss, that is, in a framework by which loads are carried horizontally from joint to joint by means of chord and web systems and finally delivered vertically upon the abutments. This is the true function of a truss; and although Palladio, Long, Howe, and Pratt had already designed and built such structures, they had never known what stresses the members were subjected to, and did as all engineers and builders did in those days—dimensioned their parts in accordance with such experience and judgment as they could bring to bear upon the problem. For instance, the vertical tie-rods in the Howe truss were at first made of the same size from end to end. Mr. Whipple's work is preserved in a small book of 120 pp. entitled "A Work on Bridge Building, consisting of Two Essays, the one elementary and general, the other giving Original Plans and Practical Details for Iron and Wooden Bridges. By S. Whipple, C.E., Utica, N. Y., 1847."

This book had been published three or four years when Hermann Haupt wrote his work on bridges. Apparently, Mr. Haupt had never seen a copy of it, since he claims his work as original also, and there is no internal evidence that he had seen Whipple's book. His methods of analysis are much cruder than Mr. Whipple's and far less complete.

The theory of the stone arch, and of arch and suspension-bridges under fixed or uniform loads, was early developed, but the true theory of truss action seems to have originated with Mr. Whipple. His manuscript was written in 1846. See *Development of the Iron Bridge*, by S. Whipple in *R. R. Gazette*, April 19, 1889; and Discussion by A. P. Boller in *Transactions Am. Soc. Civ. Engrs.*, Vol. XXV, p. 362. Also *American R. R. Bridges*, by Theodore Cooper, *Trans. Am. Soc. Civ. Engrs.*, Vol. XXI, p. 1.

intersection, horizontal-chord truss is sometimes called the Murphy-Whipple truss.

In 1861 Mr. J. H. Linville first used wide-forged eye-bars and wrought-iron posts in the web system. He still retained the cast-iron upper chord.

To Messrs. Whipple, Murphy, and Linville, therefore, is largely due the credit for establishing in this country the distinctive practice of eye-bar and pin connections which are still used here on all long-span iron bridges.

From 1865 to 1880 a great many railroad and highway bridges were built under patents granted to Mr. S. S. Post, all being of the style shown in Fig. 12. This truss is known as the Post truss, its distinctive

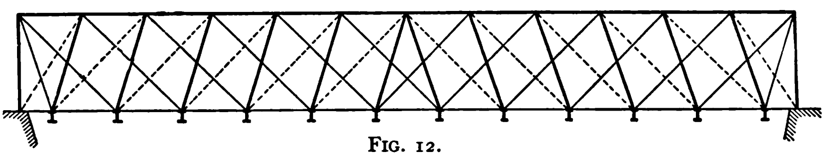


FIG. 12.

feature being that the web struts, instead of standing vertically, have a horizontal run of one-half a panel length, while the ties have a horizontal run of one and one-half panel lengths. The theoretical economy from this arrangement is now thought to be offset by corresponding practical disadvantages and the truss is no longer built.

Since this historical account treats only of truss forms, no mention is made of the early cast-iron arch-bridges, and of iron-link and wire suspension-bridges, many kinds of which, of long spans, preceded the introduction of the truss proper. In fact, in England and on the Continent the truss developed out of a combination of the arch and suspension systems, cast-iron being used in the arched upper chord, and wrought-iron links in the curved lower chord, the two being rigidly held with vertical struts and diagonal tie-rods.

Since about 1870 cast-iron has been entirely abandoned in America in the construction of railroad bridges, and since about 1880 in highway bridges as well.

The favorite style of truss now for moderate spans for all purposes is the Pratt truss (Fig. 1). It was patented in 1844 by Thomas W. and Caleb Pratt as a combination wood and iron bridge. It was a variation

from the Howe truss in that the diagonals were of iron and used in tension, while the verticals were struts and were made of timber. It never became a popular style until wrought-iron came to be used exclusively in bridge construction, when it was found to have advantage over all other forms. It will be fully developed in the body of this work.*

In closing this short account of the development of the idea of the simple truss, it should be said that only within the last forty or forty-five years have the mathematical principles governing the distribution of stresses in a truss been generally understood, and for a still shorter period has the actual strength of full-sized members and joints been even approximately known. All the earlier examples of bridge construction were designed and executed by carpenters and mechanics wholly ignorant either of the values of the stresses or of the strength of the parts, except as experience had educated their judgments of what would probably serve the purpose. They are deserving, therefore, of high honor for the great works they were able to build without any of those scientific aids now offered to every student in our numerous engineering schools.

* No historical account of American iron bridges would be complete without some notice of the efforts of Thomas Paine to introduce long cast-iron arch-bridges of low rise. As early as 1786 he advocated the use of cast-iron for long arch-bridges, with a rise of about one-twentieth of the span; and he had such arches made and tested at his own expense 90 feet in length, as models for a 400-foot span which he urged Congress to build as an example to educate the public. The Academy of Sciences of Paris reported favorably on his design for this length of span, but his model was sold for debt and afterwards used in England. He never took out a patent, his object being purely benevolent.

CHAPTER II

ELEMENTS OF THE ANALYSIS OF FRAMED STRUCTURES

SECTION I.—DEFINITIONS

18. **Forces** are *concurrent* when their lines of action meet in a point; *non-concurrent* when their lines of action do not so meet.

Forces may also be *coplanar*, that is, lying in the same plane; or *non-coplanar*, lying in different planes. Coplanar forces only will be here considered.

A force is fully defined when its *amount*, its *direction*, and its *position* are known.

19. **The Moment of a Force** about a point is its tendency to produce rotation about that point. It is measured by the product of the magnitude of the force into the perpendicular dropped from the point upon the line of action of the force.

20. **A Couple** is a pair of equal and opposite forces having different lines of action.

21. **The Resultant** of a system of forces is a single force which will replace that system as regards its effect upon the state of motion of the body acted upon. A force equal and opposite to the resultant will balance the resultant and therefore the original system. In the single case where the system reduces to a couple no one force will replace the system.

22. **Equilibrium.**—A system of forces acting upon a body is in equilibrium, or balanced, when the *state of motion* of the body is not thereby changed; e.g., a body at rest or moving at a uniform velocity is being acted upon by a balanced system of forces. The body also is said to be in equilibrium.

As we distinguish two kinds of motion, translation and rotation, so we may distinguish two kinds of equilibrium, equilibrium of translation and equilibrium of rotation. A body to be in complete equilibrium must be so in both these senses, and one does not imply the other.

For equilibrium of translation the resultant must equal zero. For equilibrium of rotation the sum of the moments of the forces about any point must equal zero.

SECTION II.—RESULTANT AND EQUILIBRIUM OF CONCURRENT FORCES

23. **Algebraic Method.**—Let P_1, . . . P_4, Fig. 1, be a system of concurrent forces applied at O. Resolve each force into horizontal and vertical components (any two directions at right angles will do as well), or along the axis of X and the axis of Y, respectively. These components form a system equivalent to the given system.

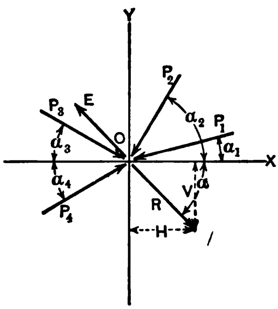


FIG. 1.

Let ΣH be the algebraic sum of the horizontal components, H, in the figure, and ΣV that of the vertical components, V, in the figure. The resultant R, of the forces $P_1 - P_4$, is evidently equal in amount to $(\overline{\Sigma H^2} + \overline{\Sigma V^2})^{\frac{1}{2}}$. The tangent of the angle a, between R and the axis of X, is given by $\Sigma V / \Sigma H$. Thus R is determined in amount and direction. Its point of application is at O, whence it becomes fully known.

For equilibrium, R must be zero, or, $(\overline{\Sigma H^2} + \overline{\Sigma V^2})^{\frac{1}{2}} = 0$, which requires that

$$\Sigma H = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\Sigma V = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two equations express the conditions which must exist in order to have equilibrium in any concurrent system of forces.

The *equilibrant*, E, is a force equal in magnitude to the resultant R, but acting in the opposite direction, as shown in Fig. 1. It is a single force which will hold in equilibrium the given system.

24. **Graphical Method.**—*Two Forces.*—Let P_1 and P_2, Fig. 2 (a), be two forces applied at point O. By the principle of the parallelogram of forces, R is their resultant in amount and direction. The same result is obtained by the use of the force triangle, Figs. (b) and (c). The order in which the forces are laid off in the figures is immaterial,

as each force will evidently have the same effect upon the final position of a point in following around the figure, in whatever part of the path the force occurs. It is necessary, however, to draw each force in its true direction.

The force E, necessary to hold the system in equilibrium, will be equal in amount but opposite in direction to R, as shown in Fig. 2 (*d*).

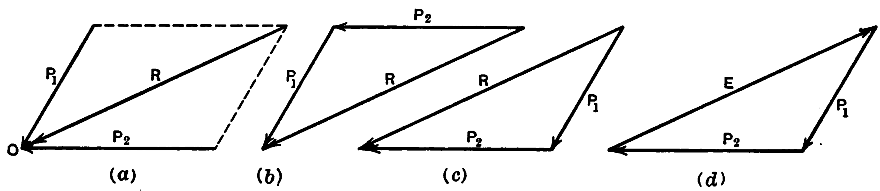


FIG. 2.

The operation of combining two forces, P_1 and P_2, into a single force R, is called the ***composition of forces.*** The reverse operation, that of dividing a force into two equivalent forces, is called the ***resolution of forces.***

25. *Any Number of Forces.*—Let $P_1 - P_4$, Fig. 3, be a system of concurrent forces applied at point O. The resultant R of the system may be found by successive applications of the triangle of force as

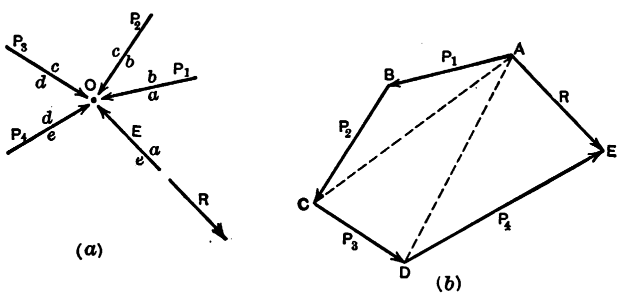


FIG. 3.

given in Art. 24. The process is as follows: Lay off $A\,B$, Fig. (*b*), equal by scale to P_1 and having the same direction, then from B lay off $B\,C$, equal and parallel to P_2. By the principle of the triangle of the forces, $A\,C$ is the resultant of P_1 and P_2 in amount and direction. Similarly by laying off $C\,D$ and $D\,E$ equal and parallel to P_3 and P_4, respectively, we have $A\,D$ equal to the resultant of P_1, P_2 and P_3, and

finally, $A\ E$ equal to the resultant of the given system. Its point of application is at O. As in the triangle of forces, the order in which the forces are taken is immaterial so long as the proper direction is used.

A figure such as Fig. 3 (*b*) is called a *Force Polygon,* and Fig. 3 (*a*) is called a *Force* or *Space Diagram.* In a force diagram, a force is represented by a line parallel to the action line of the force in question. The arrow shows the direction in which the force acts. In the force polygon, a force is represented to a certain scale by a line parallel to the action line of the force.

The force notation used in the graphical work to follow will be as shown in Fig. 3. Forces are designated in the force diagram by small letters placed on each side of the action line of the force. In the force polygon corresponding capital letters are placed at each end of the line representing the magnitude of the force. By taking consecutive letters for adjacent forces, only one letter need be placed at each apex of the force polygon.

Since R is the resultant of P_1–P_4, if a force E is applied at O, equal and opposite to R, the forces P_1–P_4, E, will form a balanced system. This is seen to be true from the force polygon also, for since $E\ A$ is equal and parallel to E, the resultant of the system P_1–P_4, E, is zero, and therefore we have equilibrium of translation. Equilibrium of rotation is not in question, the forces being concurrent. Expressed graphically, the only condition necessary for equilibrium of a system of concurrent forces is that their force polygon must close.

To sum up, we may state that the resultant of a system of concurrent forces is given in amount and direction by the closing line of the force polygon, this closing line being drawn from the origin to the end of the last force; and the force necessary to balance the given system is given by this same closing line drawn in the opposite direction.

Any system of concurrent forces, with the force polygon beginning at A and ending at E, would be equivalent to the given system, and if beginning at E and ending at A, would balance the system. Moreover, each of the forces in a closed polygon is equal and opposite to the resultant of all the other forces.

26. Solution of Problems in Equilibrium of Concurrent Forces.—*General Method of Procedure.*—Problems in the equilibrium of concurrent forces, part of which are unknown, may in certain cases, be

solved by the principles given in the preceding articles. The number of unknowns which can be determined is limited to two, for the algebraic method of Art. 23 furnishes only two independent equations, and the graphical method of Arts. 24 and 25, makes possible the determination of only two unknowns by the closure of a force polygon, the length and direction of the closing line. If more than two unknowns are in question the problem is indeterminate and a solution is not possible. The two unknowns which can be determined are the amount and direction of one force, the amounts or directions of two forces, or the amount of one and the direction of the other.

In general the first thing to be done in the solution of problems in equilibrium is to draw a sketch showing all that is known of the forces and lines of action. From the sketch find which forces or lines of action are unknown, from which decide whether the problem is determinate or not.

In applying the algebraic method, substitute the known and unknown forces, etc., in eqs. (1) and (2) of Art. 23. The solution of the resulting equations will give the values of the unknowns in terms of known forces. In applying the graphical method, complete the force polygon as far as the known data will allow. Then, subject to the condition that for equilibrium, the force polygon must close, complete the polygon with lines parallel to known action lines of unknown forces. The desired values are then given by the polygon.

Problems in equilibrium of concurrent forces capable of solution by the methods given in the preceding articles and which are of sufficient importance to require special consideration, may be classed under two heads as follows:

Given a system of concurrent forces, in equilibrium, (*a*) all known except one, which is wholly unknown, (*b*) all known in line of action, but two are unknown in amount. In the following articles, a general solution of these problems will be given.

27. *Algebraic Solution*—(*a*) Given a system of concurrent forces in equilibrium, all known but one which is wholly unknown.

Let P_1–P_4, Fig. 4, be the given system, P_4 being wholly unknown. Two unknowns are to be determined, the amount and position of P_4. Assume P_4 to act as shown. Required P_4 and α_4.

Substituting known values in eqs. (1) and (2), of Art. 23, considering forces acting upward and to the right as positive we have

From $\Sigma H = 0$

$$+ P_1 \cos \alpha_1 + P_2 \cos \alpha_2 - P_3 \cos \alpha_3 - P_4 \cos \alpha_4 = 0.$$

Whence

$$\text{Hor. comp. } P_4 = + P_4 \cos \alpha_4$$
$$= + P_1 \cos \alpha_1 + P_2 \cos \alpha_2 - P_3 \cos \alpha_3 \quad . \quad . \quad . \quad (3)$$

From $\Sigma V = 0$

$$+ P_1 \sin \alpha_1 - P_2 \sin \alpha_2 - P_3 \sin \alpha_3 + P_4 \sin \alpha_4 = 0.$$

Whence

$$\text{Vert. comp. } P_4 = + P_4 \sin \alpha_4$$
$$= - P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 \quad . \quad . \quad . \quad (4)$$

If the correct direction was assumed for P_4 in Fig. 4, the sign of the results in eqs. (3) and (4) will be positive. A negative result shows

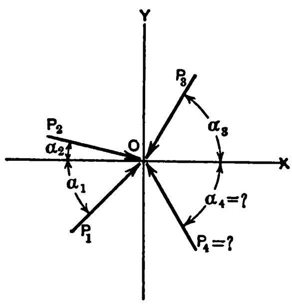


FIG. 4.

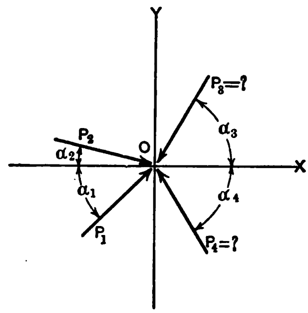


FIG. 5.

that the direction should be reversed. Then, having the directions of the components of P_4, it is easy to find the direction of the required force.

The amount of P_4 is given by the equation

$$P_4 = [(\text{Hor. comp. } P_4)^2 + (\text{Vert. comp. } P_4)^2]^{\frac{1}{2}} \quad . \quad . \quad (5)$$

The angle which P_4 makes with the axis of X is

$$\alpha_4 = \tan^{-1} \frac{\text{Vert. comp. } P_4}{\text{Hor. comp. } P_4}. \quad . \quad . \quad . \quad . \quad (6)$$

P_4 and α_4 are thus known and the required force is completely determined.

(*b*) Given a system of concurrent forces in equilibrium, all known in line of action, but two unknown in amount.

Let P_1–P_4, Fig. 5, be the given system, P_1 and P_2 completely known, P_3 and P_4 known only in line of action. Two unknowns are to be determined, the amounts of P_3 and P_4.

Assume P_3 and P_4 to act in the direction shown by the arrows. Substituting in eqs. (1) and (2) of Art. 23, considering forces acting upward and to the right as positive gives

From $\Sigma H = 0$

$$- P_3 \cos \alpha_3 - P_4 \cos \alpha_4 + P_1 \cos \alpha_1 + P_2 \cos \alpha_2 = 0 \quad . \quad (7)$$

From $\Sigma V = 0$

$$- P_3 \sin \alpha_3 + P_4 \sin \alpha_4 + P_1 \sin \alpha_1 - P_2 \sin \alpha_2 = 0 \quad . \quad (8)$$

We have here two independent equations involving the unknowns P_3 and P_4. Solving eqs. (7) and (8) simultaneously gives

$$P_4 = \frac{+ P_1 \sin (\alpha_3 - \alpha_1) + P_2 \sin (\alpha_3 + \alpha_2)}{\sin (\alpha_3 + \alpha_4)} \quad . \quad . \quad . \quad (9)$$

$$P_3 = \frac{+ P_1 \sin (\alpha_4 + \alpha_1) + P_2 \sin (\alpha_4 - \alpha_2)}{\sin (\alpha_3 + \alpha_4)} \quad . \quad . \quad . \quad (10)$$

As before, the sign of the result will show whether the correct directions are assumed for P_3 and P_4.

28. *Graphical Solution.*—(*a*) Given a system of concurrent forces in equilibrium, all known except one which is wholly unknown.

Let P_1–P_4, Fig. 6 (*a*), be the given system, P_4 unknown. Two unknowns are to be determined, the amount and direction of P_4. This

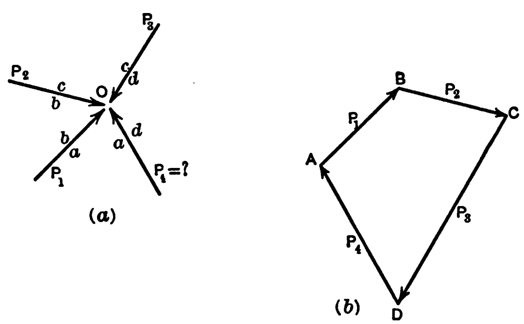


FIG. 6.

is the same problem as given in Art. 25, and consists in finding the equilibrant of a set of forces.

To find the unknowns, construct the force polygon as far as the known data will allow. This gives the load-line $A\ B\ C\ D$ of Fig. (*b*).

For equilibrium the force polygon must close. A line drawn from D to A will give in magnitude and direction the required force. A line drawn through O, Fig. (a), parallel to $D\,A$, will give the position of P_4 in the force diagram. The force acts as shown by the arrow. P_4 is thus completely determined.

(b) Given a system of forces, all known in line of action, but two unknown in amount.

Let P_1–P_4, Fig. 7 (a), be the given system, P_3 and P_4 known only in lines of action, as shown. Two unknowns are to be determined, the amounts of P_3 and P_4.

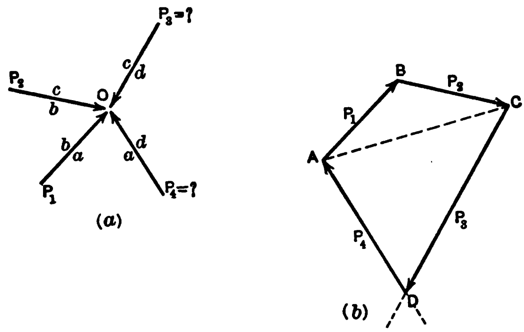


FIG. 7.

Draw the force polygon, Fig. (b), as far as the known data will allow. This gives the load-line $A\,B\,C$. For equilibrium the force polygon must close. From the conditions of the problem, the closing forces have action lines parallel to $c\,d$ and $a\,d$ of Fig. (a). To complete the force polygon, draw from C a line parallel to $c\,d$, and from A a line parallel to $a\,d$. The point D at which these lines meet will determine the magnitude of the required forces. Thus $D\,A$ acting from D to A, gives the magnitude and direction of P_4, and $C\,D$, acting from C to D gives the magnitude and direction of P_3. The forces are thus completely determined.

It is immaterial in what order the forces are drawn, but the arrangement shown conforms to the system of notation adopted. The same result could be obtained by taking $A\,C$, the resultant of P_1 and P_2, and resolving this force into components parallel to $c\,d$ and $a\,d$ by means of a force triangle as explained in Art. 24.

SECTION III.—RESULTANT AND EQUILIBRIUM OF NON-CONCURRENT FORCES

29. Algebraic Method.—Let P_1–P_3, of Fig. 8, be a system of non-concurrent forces. Required their resultant and conditions of equilibrium.

Resolve the forces into vertical and horizontal components, as in the case of concurrent forces. The amount of the resultant is given by

$$R = (\overline{\Sigma H}^2 + \overline{\Sigma V}^2)^{\frac{1}{2}}.$$

Its line of action is found by placing its moment about O equal to the sum of the moments of the given forces, or

$$R\,a = P_1\,a_1 + P_2\,a_2 - P_3\,a_3 = \Sigma M.$$

To cause equilibrium requires the application of a force E (the equilibrant), equal and opposite to R, and having the same line of action.

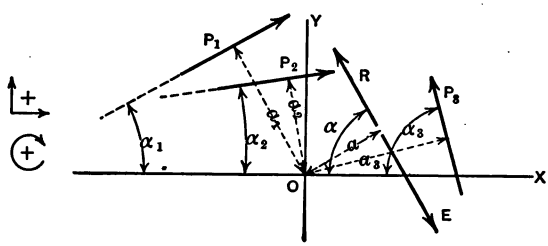


FIG. 8.

The resultant of the system P_1, P_2, P_3, E, will be zero and the sum of the moments of these forces about O will be zero. The former condition gives equilibrium of translation and requires that

$$\Sigma H = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the latter gives equilibrium of rotation and requires that

$$\Sigma M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (1), (2), and (3) are the three equations of equilibrium of non-concurrent forces. These three equations hold true for any system of non-concurrent forces in equilibrium. In problems pertain-

ing to such a system we can therefore in general determine three unknowns. As the forces are, as a rule, known in position and direction from other considerations, the unknowns are usually the amounts of three forces. If, however, the three forces themselves meet in a point they are indeterminate; for, since the system is by supposition in equilibrium, the resultant of the known forces must pass through the same point, and the system is essentially a concurrent system. As we have then but two independent equations, a solution of three unknowns is not possible.

In many cases it is convenient to use three moment equations taking a new moment centre each time instead of two resolution equations and one moment equation. Eqs. (1) and (2) are then no longer independent. In the case of parallel forces the number of independent equations reduces to two, since the assumption of parallel forces is equivalent to eqs. (1) or (2) (either ΣH or ΣV will be zero). In this case we then have

$$\Sigma H \text{ or } \Sigma V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\Sigma M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

from which two unknowns may be determined.

30. Graphical Method.—Let P_1–P_4, Fig. 9, be a system of non-concurrent forces. Required their resultant and conditions of equilibrium.

By means of the principle of the triangle of forces as given in Art. 24, combine P_1 and P_2, getting their resultant R_1, then this resultant with

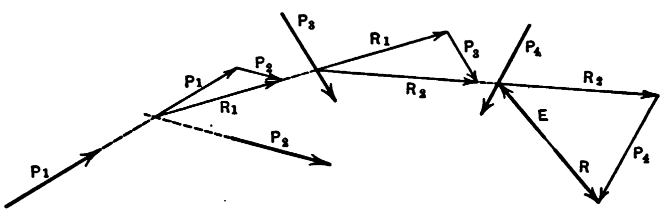


FIG. 9.

P_3 getting R_2, and finally R_2 and P_4, getting R, the resultant of the entire system in amount, direction and position.

To save drawing the separate force triangles we may draw a force polygon as in Fig. 10 (b). The lines $A\,C$, $A\,D$ and $A\,E$ will be the

resultants R_1, R_2, and R, respectively, of Fig. 9, in amount and direction. Their position in Fig. (*a*) can be found by drawing from point 1, the intersection P_1 and P_2, a line 1–2 parallel to R_1, Then from the intersection of 1–2 and P_3, draw 2–3 parallel to R_2. Finally from the intersection of 2–3 and P_4, draw a line parallel to R. This will be the line of action of R, whose amount and direction are given in the force polygon. Thus R will be completely determined.

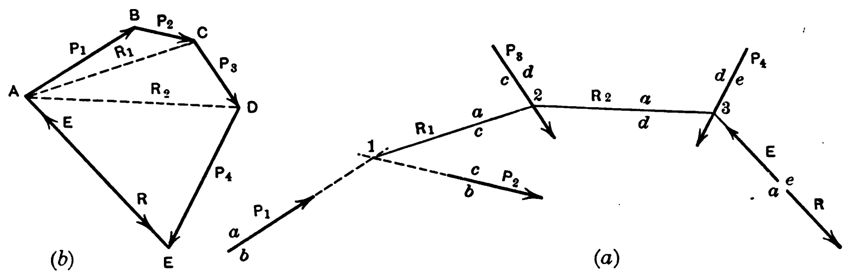


FIG. 10.

The figure 1–2–3 is called an *equilibrium polygon*, and 1–2, 2–3 are its *segments*. The point A is called the *pole*; R_1, R_2, and R are *rays* of the force polygon; and $A\ B\ C\ D\ E$ is called the *load-line*.

To cause equilibrium there must be applied a force E, equal and opposite to R, and applied along the same line. Graphically, then, for equilibrium of non-concurrent forces, the force polygon must close, giving equilibrium of translation, and the last force must coincide with the last segment of the equilibrium polygon, giving equilibrium of rotation. It is important to remember that each segment of an equilibrium polygon is the line of action of the resultant of all the forces to the left of that segment, and if the system is in equilibrium, it is as well the line of action of the resultant of all the forces to the right of that segment.

The notation used for the equilibrium polygon is shown in Fig. 10 (*a*). As the segments of the equilibrium polygon are the resultants of the forces at the left, we can use the same notation as used for forces. Thus in Fig. (*a*) the small letters *a–c*, *a–d*, etc., are placed on each side of the line of action of the various resultants, and in Fig. (*b*) the corresponding capital letters are placed at the ends of the magnitudes of the resultants.

31. *Forces nearly parallel* require some modification of the method of Art. 30 in order to make better intersections between the segments of the equilibrium polygon. Given the forces P_1–P_4, Fig. 11 (*a*), required the resultant.

The force polygon, Fig. (*b*), gives their resultant $A\,E$ in amount and direction; $E\,A$ is a force which will balance the given system. Resolve this force into any two components, P' and P'', by drawing the force triangle $E\,O\,A$, O being so chosen as to give fair angles with all the other forces. The forces P' and P'', together with the given forces, P_1–P_4, will form a system in equilibrium, provided they are inserted in

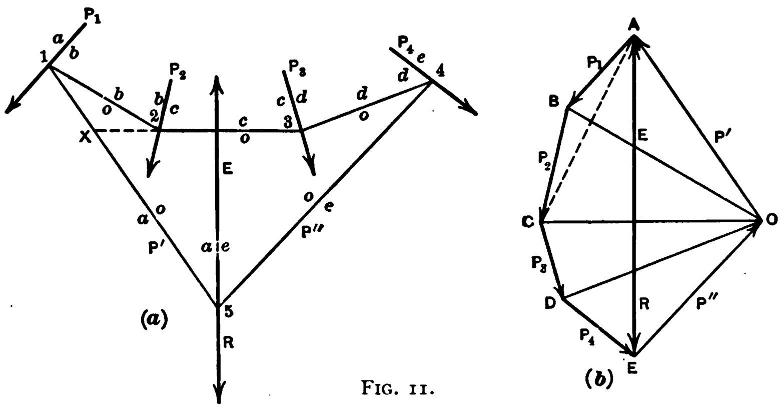


FIG. 11.

Fig. 11 (*a*), in the proper positions. The position of one of them, P', may be chosen at will, and the other will be given by the last segment of the equilibrium polygon constructed for the forces P', P_1–P_4, with O as the pole of the force polygon. Choose the position of P' at any point, as 1. Complete the equilibrium polygon, 1–2, 2–3, 3–4. This is done in exactly the same way as in the preceding article. Point 4 will be a point on the line of action of P'', the other arbitrary force. Draw 4–5 parallel to P'' of Fig. (*b*). The last force thus coinciding with the last segment, and the force polygon closing, equilibrium is assured. The intersection, 5, of P' and P'' is a point on the resultant of these two forces, and since $E\,A$ is this resultant in amount and direction, it is therefore fully known. It is represented by E in Fig. (*a*), and since it balances P_1–P_4, an equal and opposite force, R, is the required resultant of the given system, in amount, direction, and position.

Since the resultant of all the forces up to any segment applied along that segment will balance the remaining forces, it follows that the intersection of ***any two segments*** of an equilibrium polygon is a point on the resultant of the intermediate forces. Thus the point X is a point on the resultant of P_1 and P_2; the line $A\ C$ in the force polygon gives the amount and direction of this resultant.

32. *Parallel Forces.*—The method is the same as for forces nearly parallel.

Let P_1–P_4, Fig. 12, be a system of parallel forces. Required the resultant and conditions of equilibrium. Figs. (*a*) and (*b*) show all the necessary construction.

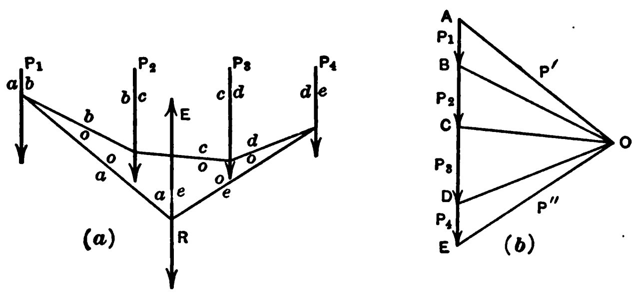


FIG. 12.

33. Solution of Problems in the Equilibrium of Non-Concurrent Forces.—*General Method of Procedure.*—Problems in the equilibrium of non-concurrent forces involving not more than three unknowns may be solved by means of the principles given in Arts. 29–32. The number of unknowns which can be determined is limited to three, for the algebraic method of Art. 29 furnishes only three independent equations. Also the closing of the forces and equilibrium polygons in the graphical method of Arts. 30 and 31 makes possible the determination of three unknowns, represented by the length, position, and direction of the closing line. If more than three unknowns are in question the problem is indeterminate.

The three unknowns usually desired in problems relating to framed structures may be classed under a few general cases. Only those cases will be considered which are likely to arise in practice. These cases are the following:

Given a system of non-concurrent forces in equilibrium:

(*a*) All known but one, which is wholly unknown.

(*b*) All known but two, one of these known only in line of action, the other known only in point of application. A special case is when both are known in line of action only.

(*c*) All known in line of action, but three unknown in amount.

When the forces involved are parallel, the number of independent equations reduces to two, as given in eqs. (4) and (5) of Art. 29. It is therefore possible to determine but two unknowns. The problems to be solved involving parallel forces can be classed under the following cases:

(*a*) All known but one, which is completely unknown.

(*b*) All known in line of action, but two unknown in amount.

In applying the algebraic method to the solution of a given problem, it is possible to apply the three independent equilibrium equations in several ways. We can use two resolutions and one moment equation, or one resolution and two moment equations, or three moment equations. In solving a given problem a sketch should first be drawn showing all known forces and lines of action, and then the unknowns indicated whose values are required. From a study of this sketch it is usually possible to determine by inspection which independent equations will give a solution with the least amount of work. Then substitute known values in the equations selected and solve for the unknowns, which can thus be obtained in terms of known quantities.

In applying the graphical method it will be necessary to draw a force and equilibrium polygon, for equilibrium of rotation as well as translation is to be obtained. Then, by closing the force and equilibrium polygon, subject to the conditions of the given problem, the unknowns can be determined. In the following articles a general solution, both algebraical and graphical, will be given for the cases mentioned above.

34. *Algebraic Solution.*—(*a*) Given a system of non-concurrent forces in equilibrium, all known but one, which is completely unknown. Required the unknown force.

Let P_1–P_4, Fig. 13, be the given system, P_4 wholly unknown. Since the system is in equilibrium, P_4 must balance the remaining forces. Choose any convenient axes, X and Y at right angles. Let forces act-

ing upward and to the right be taken as positive. Assume P_4 to be divided into its components parallel to X and Y, taken as acting in positive directions. For equilibrium of translation ΣH and ΣV must be zero.

Therefore from $\Sigma H = 0$

$$+ P_1 \cos \alpha_1 + P_2 \cos \alpha_2 - P_3 \cos \alpha_3 - \text{Hor. comp. } P_4 = 0$$

$$\text{Hor. comp. } P_4 = + P_1 \cos \alpha_1 + P_2 \cos \alpha_2 - P_3 \cos \alpha_3 \quad . \quad (6)$$

from $\Sigma V = 0$

$$- P_1 \sin \alpha_1 - P_2 \sin \alpha_2 - P_3 \sin \alpha_3 + \text{Vert. comp. } P_4 = 0$$

$$\text{Vert. comp. } P_4 = + P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3. \quad . \quad (7)$$

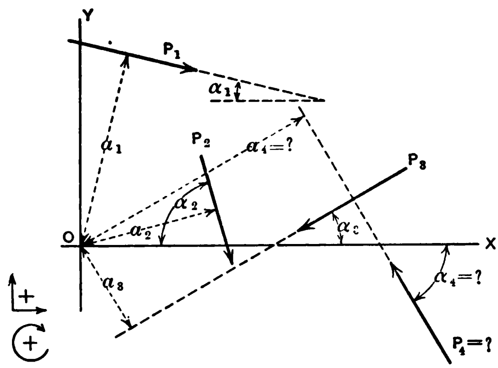


FIG. 13.

The sign of the result in eqs. (6) and (7) will show whether the correct direction was assumed.

The amount of the force P_4 will be

$$P_4 = [(\text{Hor. comp. } P_4)^2 + (\text{Vert. comp. } P_4)^2]^{\frac{1}{2}}. \quad . \quad . \quad (8)$$

The angle which the line of action of P_4 makes with the axis of X is

$$\alpha_4 = \tan^{-1} \frac{\text{Vert. comp. } P_4}{\text{Hor. comp. } P_4}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The signs of the vertical and horizontal components of P_4 will show how this line slopes. If both are positive or negative, the line of action of P_4 slopes upward and to the right; if one is positive and the other negative, the line slopes downward and to the right.

The position of P_4 can be determined from the condition that

$\Sigma M = 0$. Considering moments clockwise as positive, the origin being taken at O, gives

$$P_1 a_1 + P_2 a_2 + P_3 a_3 - P_4 a_4 = 0, \quad . \quad . \quad . \quad . \quad (10)$$

from which

$$a_4 = \frac{P_1 a_1 + P_2 a_2 + P_3 a_3}{P_4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The direction of P_4 can be determined from eq. (10), for it must be such as to balance the moments of the other forces.

The position of P_4 can also be determined by finding the point at which the line of action cuts the X or Y axis. Suppose it cuts the X axis, a distance x from the origin. Resolve the force into its vertical and horizontal components, considered as applied at the point where the action line cuts the axis of X. Taking moments about the origin we have

$$+P_1 a_1 + P_2 a_2 + P_3 a_3 - \text{Hor. comp. } P_4 \times 0 - \text{Vert. comp. } P_4 \times x = 0$$

$$x = \frac{P_1 a_1 + P_2 a_2 + P_3 a_3}{\text{Vert. comp. } P_4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The sign of x shows whether it lies to right or left of the origin, and, as before, the direction of the line of action is given in eq. (10).

(*b*) Given a system of non-concurrent forces in equilibrium, all known but two, one known only in line of action, the other only in point of application. Required the unknown forces.

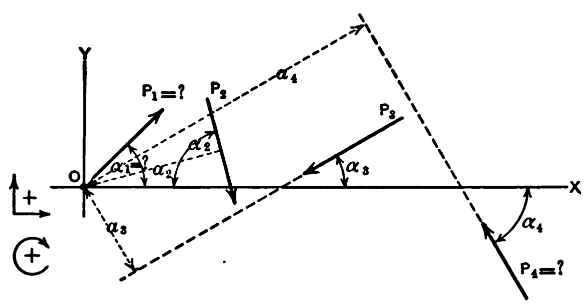


FIG. 14.

Let P_1–P_4, Fig. 14, be the given system. P_1 is known only in point of application, which is at point O, and P_4 is known only in line of action, which is given by the dotted line. Since the system is in

equilibrium, the unknown forces must balance the known forces. Assume the direction of P_4 to be as shown by the arrow. Choose a pair of axes, X and Y, with the origin at the known point of application of P_1. A moment equation about point O will eliminate P_1, and the amount of P_4 will then be the only unknown.

From $\Sigma M = 0$

$$+ P_2 a_2 + P_3 a_3 - P_4 a_4 = 0$$

$$P_4 = \frac{P_2 a_2 + P_3 a_3}{a_4}. \qquad (13)$$

The sign of the result will show whether the correct direction was assumed.

The force, P_1, can now be determined by resolution equations. Assume P_1 to be divided into components parallel to the X and Y axes and assume that they act in positive directions. For equilibrium of translation ΣH and ΣV must be zero.

Therefore ($\Sigma H = 0$)

$$+ P_2 \cos \alpha_2 - P_3 \cos \alpha_3 - P_4 \cos \alpha_4 + \text{Hor. comp. } P_1 = 0$$

$$\text{Hor. comp. } P_1 = - P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + P_4 \cos \alpha_4, \qquad (14)$$

also ($\Sigma V = 0$)

$$- P_2 \sin \alpha_2 - P_3 \sin \alpha_3 + P_4 \sin \alpha_4 + \text{Vert. comp. } P_1 = 0$$

$$\text{Vert. comp. } P_1 = + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 - P_4 \sin \alpha_4. \qquad (15)$$

The signs of the results will show whether the correct directions were assumed.

The amount of P_1 will be

$$P_1 = [(\text{Hor. comp. } P_1)^2 + (\text{Vert. comp. } P_1)^2]^{\frac{1}{2}}. \qquad (16)$$

The angle which P_1 makes with the axis of X is

$$\alpha_1 = \tan^{-1} \frac{\text{Vert. comp. } P_1}{\text{Hor. comp. } P_1}. \qquad (17)$$

The slope of the line of action is determined as in the preceding case. The point of application is at the origin. The unknown forces are hence completely determined.

(*c*) Given a system of non-concurrent forces in equilibrium, all known in line of action, but three unknown in amount. Required the amounts of the unknown forces.

Let P_1–P_4, Fig. 15, be the given system, P_2, P_3 and P_4 known only in lines of action, which are shown in position by the dotted lines.

The problem can be solved by taking moment equations about points so chosen as to eliminate all but one of the unknowns. This

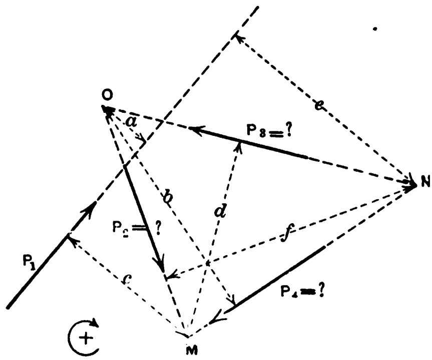


FIG. 15.

can be done by taking the moment centre at the intersection of two of the unknowns. Taking moments about O, the intersection of P_2 and P_3, assuming the unknowns to act as shown by the arrows, we have

$$-P_1 a + P_4 b = 0,$$

from which

$$P_4 = +\frac{P_1 a}{b}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Taking moments about M, the intersection of P_2 and P_4 gives

$$+P_1 c - P_3 d = 0,$$

from which

$$P_3 = +\frac{P_1 c}{d}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Again, taking moments about N, the intersection of P_3 and P_4 gives

$$+P_1 e - P_2 f = 0$$

from which

$$P_2 = +\frac{P_1 e}{f}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

In some cases, where two of the unknowns are parallel or nearly so, this method can be modified by taking two moment and one resolution

equations. The method is then a combination of the above case and the previous cases.

35. *Graphical Solution.*—(*a*) Given a system of non-concurrent forces in equilibrium, all known but one, which is completely unknown. Required the unknown force.

Let P_1–P_4, Fig. 16, be the given system, P_4 being unknown. Since the system is in equilibrium the force and equilibrium polygons must close. Drawing the force polygon of Fig. (*b*) gives the amount and direction of P_4 as given by $D\,A$. To determine the position of P_4 in Fig. (*a*), draw an equilibrium polygon with O as pole. The point of intersection of segments $o\,a$ and $o\,d$ will give a point on the line of

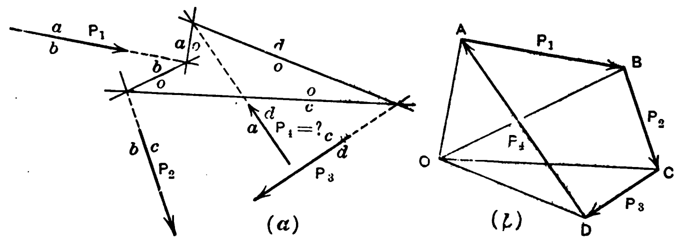


FIG. 16.

action of P_4 which can be drawn parallel to $D\,A$ of Fig. (*b*) and in the direction shown by the arrow. P_4 is thus completely known. The above case is exactly the same as treated in Art. 32.

(*b*) Given a system of non-concurrent forces in equilibrium, all known but two, one of these known only in line of action, the other only in point of application.

Let P_1–P_4, Fig. 17, be the given system. P_1 is known only in point of application, which is at O. P_4 is known only in line of action, which is shown by the dotted line. The unknown forces can be completely determined by the condition that for equilibrium the force and equilibrium polygons must close.

With the known forces draw the force polygon of Fig. (*b*) as far as possible, which gives B, C, D of the polygon. Also, draw from D a line parallel to P_4. With O as pole draw the rays $O\,B$, $O\,C$, and $O\,D$. Draw an equilibrium polygon as far as the known data will allow. Since O is the only known point on P_1, it will be necessary to start the equilibrium polygon at this point. This gives O 2 3 4, of Fig. (*a*).

The condition for equilibrium of rotation requires that the equilibrium polygon must close. O 4 of Fig. (*a*) is, therefore, the required closing line. At point 4 the segments 3 4 and O 4, and the unknown force P_4, form a system of forces in equilibrium. The force 3–4 is given in amount and direction by $O\ D$ of Fig. (*b*). The line $O\ A$, drawn parallel to 0–4, closes the force triangle $O\ A\ D$ and determines the amount and direction of P_4.

Again, at O of Fig. (*a*), force P_1 with segment O 2 and O 4 form a system of forces in equilibrium. O 2 is given in amount and direction by $O\ B$, and O 4 by $O\ A$, as found above. Thus by completing the force triangle OAB, P_1 is determined in amount and direction by $A\ B$. Then through O of Fig. (*a*) draw a line parallel to $A\ B$ of Fig. (*b*), which

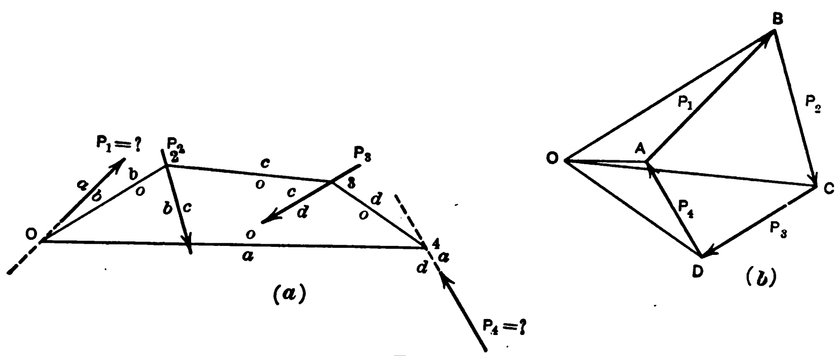


FIG. 17.

will give the line of action of P_1. P_1 and P_4 must have directions such that the arrows in the force polygon will follow continuously from the point of beginning to closing point. P_1 and P_4 are thus completely determined.

The case where the two unknowns are both known in line of action but unknown in amount often occurs. Here it is unnecessary to draw an equilibrium polygon (except where the unknowns are parallel, which will be treated in another article). It is only necessary to draw a force polygon and satisfy the condition that the force polygon closes. Let P_1–P_4, Fig. 18, be such a system. P_1 and P_4 are known only in line of action. Fig. (*a*) shows the position of the forces in space. Draw the force polygon, Fig. (*b*), as far as the known forces will allow. This gives the load-line $B\ C\ D$. Then from B and D draw lines parallel

respectively to the known action lines of P_1 and P_4, which will give $A\,B$ and $D\,A$. These lines meet at D, thus determining the amounts of the unknowns. Their directions will be as shown by the arrows. The order in which the unknowns are taken will not affect the result.

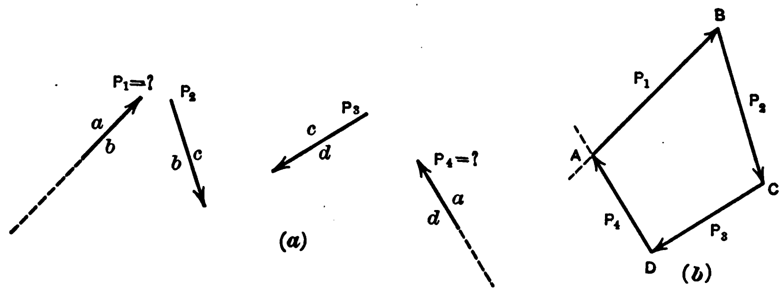


FIG. 18.

(*c*) Given a system of non-concurrent forces in equilibrium, all known in line of action but three unknown in amount. Required the amounts of the unknown forces.

Let P_1–P_5, Fig. 19, be the given system, P_1, P_2 and P_5 known only in line of action. By producing two of the unknowns to an intersection and considering the two forces replaced by their resultant, this case

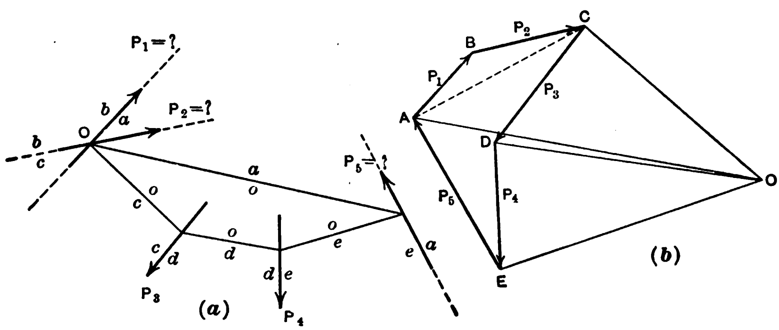


FIG. 19.

becomes a special case of case (*b*) above. The construction is as shown in Fig. 19. P_1 and P_2 intersect at O, at which point the equilibrium polygon is started as before. P_5 is given in amount and direction by $E\,A$, of Fig. (*b*). $A\,C$ gives in amount and direction, the resultant of P_1 and P_2. By drawing the force triangle $A\,B\,C$, with $A\,B$ and $B\,C$ parallel respectively to P_1 and P_2, the amounts and directions of these

forces can be obtained and their directions shown in Fig. (*a*). The unknowns are thus completely determined.

36. *Parallel Forces.*—(*a*) Given a system of parallel forces in equilibrium, all known but one which is completely unknown. Required the amount and position of the unknown force.

Algebraically.—Let P_1–P_4, Fig. 20, be the given system, P_4 wholly unknown. Since the system is in equilibrium, the unknown force must balance the remaining forces. Choose any convenient axes

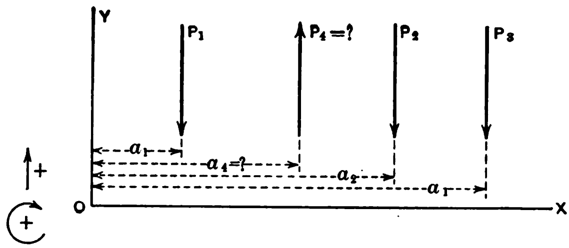


FIG. 20.

X and Y at right angles and so taken as to have one of the axes parallel to the given forces. The amount of the unknown force, P_4, may be determined by a resolution equation. Assume P_4 to act upwards.

From $\Sigma V = 0$

$$+P_4 - P_1 - P_2 - P_3 = 0$$

$$P_4 = +P_1 + P_2 + P_3. \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

The position of P_4 can be determined by a moment equation. Take O, the origin, as the moment centre, and let a_4 be the distance from O to the line of action of P_4.

From $\Sigma M = 0$

$$+P_1 a_1 + P_2 a_2 + P_3 a_3 - P_4 a_4 = 0$$

$$a_4 = \frac{+P_1 a_1 + P_2 a_2 + P_3 a_3}{P_4}. \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

The amount and position of P_4 are thus given and as its line of action is parallel to the other forces, the unknown is completely determined.

Graphically.—The graphical solution of this problem is the same as given in Art. 32.

(*b*) Given a system of non-concurrent parallel forces all known in

line of action, but two unknown in amount. Required the amounts of the unknown forces.

Algebraically.—Let P_1–P_4, Fig. 21, be the given system, P_1 and P_4 known only in line of action as shown by the dotted lines. Choose any convenient axes as X and Y. Here the axis Y is assumed so as to coincide with the line of action of P_1. The unknowns can be determined from the condition that for equilibrium of rotation $\Sigma M = 0$. The moment equations can be written so as to involve but one unknown.

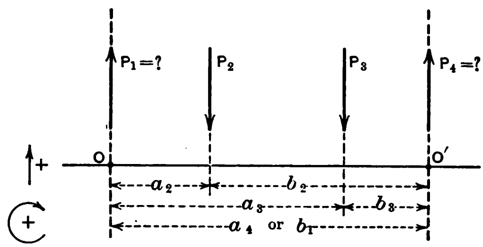


FIG. 21.

This is done by taking the moment centre on the known line of action of one of the unknown forces. Thus a moment equation about the origin O, assuming P_4 to act upward, gives

$$+ P_2 a_2 + P_3 a_3 - P_4 a_4 = 0$$

$$P_4 = \frac{+ P_2 a_2 + P_3 a_3}{a_4} \quad \dots\dots \quad (23)$$

In the same way, a moment equation about O', a point in the line of action of P_4 will give,

$$- P_3 b_3 - P_2 b_2 + P_1 b_1 = 0$$

$$P_1 = \frac{+ P_3 b_3 + P_2 b_2}{b_1}. \quad \dots\dots \quad (24)$$

The value of P_4 could have been obtained by use of a resolution equation, since for equilibrium of translation, $\Sigma V = 0$. Since P_1 is given in eq. (24), P_4 is the unknown, and we have

$$- P_2 - P_3 + P_1 + P_4 = 0$$

$$P_4 = P_2 + P_3 - P_1. \quad \dots\dots \quad (25)$$

Thus the unknowns in such a case may be obtained by the use of two

independent moment equations, or by the use of one moment and one resolution equation.

Graphically.—As above, let P_1–P_4 be the given system, P_2 and P_3 fully known, P_1 and P_4 known only in line of action. Required the amounts of the unknown forces. This is a special case of the problem

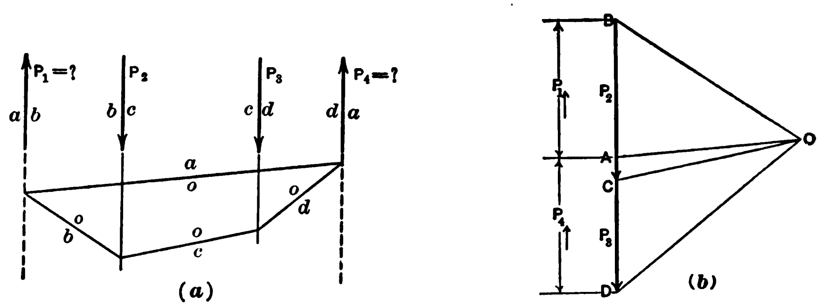


FIG. 22.

given in section (*b*), of Art. 35. The method is exactly the same. The necessary construction is shown in Fig. 22.

37. ***Problem. To Pass an Equilibrium Polygon through Three Given Points.***—Let P_1–P_5, Fig. 23, be the given forces and R, S and T the given points. Select any pole O' and draw an equilibrium polygon through one of the given points, as S. Consider the given system divided into two parts and find the resultants of each part. In Fig. (*b*), $A\,D$ is the resultant of P_1, P_2 and P_3 and $D\,F$ is the resultant of P_4 and P_5. Through R and T draw $R\,R'$ and $T\,T'$ parallel to these resultants. Draw $R'\,S$ and $S\,T'$, the closing lines of the equilibrium polygons for the two parts of the given system. In Fig. (*b*), draw $O'\,n_1$ and $O'\,n_2$ parallel respectively to $R'\,S$ and $S\,T'$. $O'\,n_1$ and $O'\,n_2$ will divide the resultants into components which are equivalent to the two systems P_1–P_3 and P_4–P_5 when applied through points R and S, and S and T respectively. The same results would evidently be obtained from any other equilibrium polygon drawn through S. The line $R\,S$ will be the closing line of an equilibrium polygon for P_1–P_3, passing through R and S. Its pole will evidently lie somewhere on a line $n_1\,O$, drawn parallel to R S. Likewise the line $n_2\,O$ drawn parallel to $S\,T$ will contain the pole of the equilibrium polygon for P_4 and P_5 passing

through S and T. Hence the pole of the equilibrium polygon of the whole system passing through R, S and T will lie at point O, the intersection of $n_1 O$ and $n_2 O$.

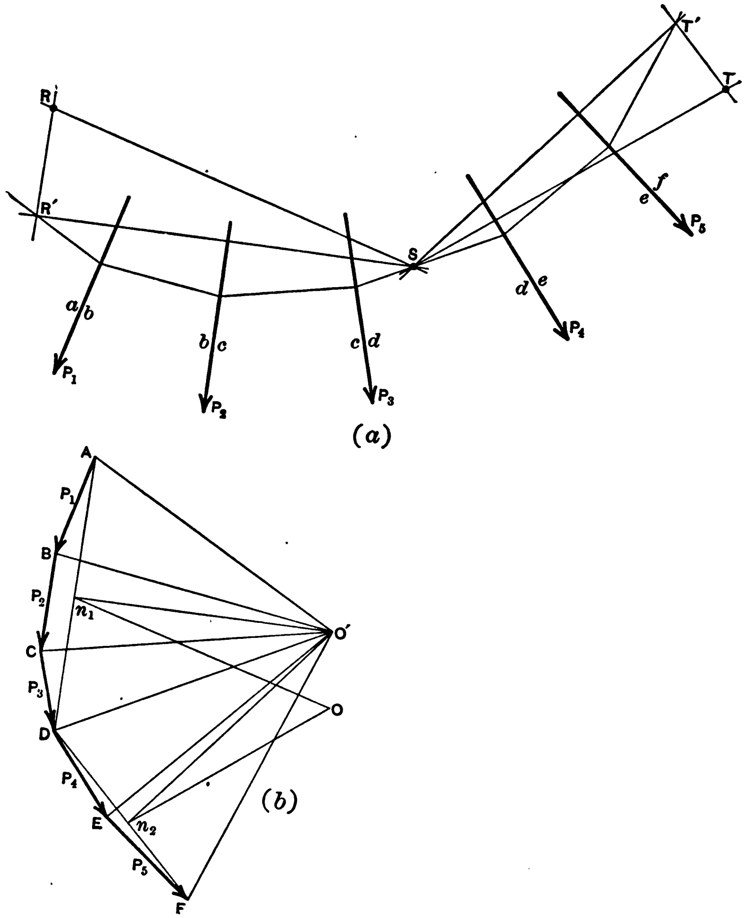


FIG. 23.

For parallel forces the same method can be used. For this case a somewhat shorter solution is given in Fig. 24. This solution depends upon the fact that the pole distances of any two equilibrium polygons, for the same system of parallel forces, are inversely proportional to the intercepts on a line parallel to the forces. In Fig. (a), R 1 2 3 4 T'

is an equilibrium polygon drawn for any selected pole distance, H, and passing through one of the points, R. y is the intercept on a line drawn through S parallel to the forces. Since $R\ T$ is the closing line for the required polygon, y' will be the intercept for the re-

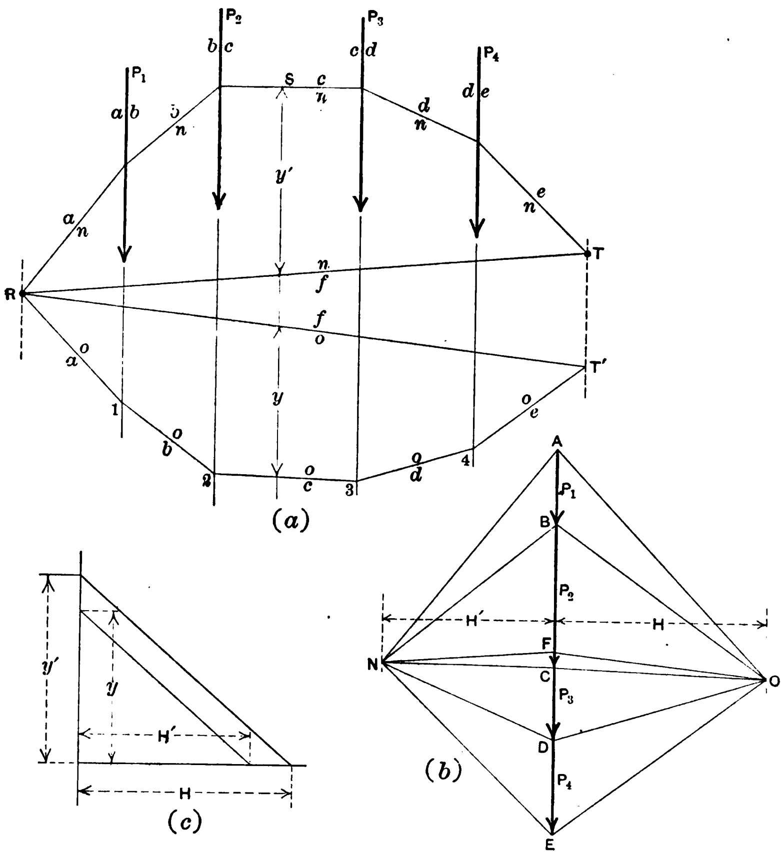


FIG. 24.

quired polygon. Then the required pole distance H' is found from the proportion $H' : H = y : y'$; or by the construction of Fig. (*c*). To find the new pole N, Fig. (*b*), draw $O\ F$ parallel to $R\ T'$ and $F\ N$ parallel to $R\ T$. The pole N is located on $F\ N$ at a distance H' from the load line.

Section IV.—Centre of Gravity, Moments, and Moments of Inertia

38. Centre of Gravity.—The centre of gravity of a body is the point through which the resultant of all the gravity forces acting on a body will pass. The problem of determining centres of gravity is therefore one in parallel forces, since the attraction of gravity, or of one body on another acts in parallel lines.

Since the sum of the moments of the gravity forces acting on the particles of a body, referred to any plane, is equal to the moment of the resultant gravity force of the whole body about the same plane, we can write (considering gravity as acting parallel to the plane in question),

$$\left.\begin{array}{l}\bar{x}\,\Sigma\,\omega = \Sigma\,\omega x\\ \bar{y}\,\Sigma\,\omega = \Sigma\,\omega y\\ \bar{z}\,\Sigma\,\omega = \Sigma\,\omega z\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x, y and z are the distances of the various particles of the body for the X, Y and Z planes respectively, $\bar{x}$, $\bar{y}$ and $\bar{z}$ the distances of the centre of gravity of the entire body from the same planes, ω is the weight of each particle and $\Sigma\omega$ the total weight of the body.

To find the coordinate of the centre of gravity we have from (1)

$$\left.\begin{array}{l}\bar{x} = \dfrac{\Sigma\,\omega x}{\Sigma\,\omega}\\[2ex] \bar{y} = \dfrac{\Sigma\,\omega y}{\Sigma\,\omega}\\[2ex] \bar{z} = \dfrac{\Sigma\,\omega z}{\Sigma\,\omega}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

39. *Centre of Gravity of Areas and Lines.*—It is frequently convenient to speak of the centre of gravity of bodies which have no weight, such as areas or lines. Here a portion of the area of the surface or of the length of the line can be used as a unit, in place of ω. In the work to follow particular attention will be given to areas.

Any area whose centre of gravity is desired, can be divided into single areas whose gravity centres are easily obtained, as squares, rectangles, or triangles. Then by applying at their centres, forces in proportion to the partial areas, a system of parallel forces is obtained

whose resultant will give the position of the centre of gravity with respect to one axis of coordinates. Then by repeating the process with respect to another coordinate axis a centre of gravity is obtained with respect to the second axis. The intersection of these two gravity lines will give the centre of gravity of the area for all axes. It is usual to choose axes at right angles to each other. The axes here chosen will be the axes of X and Y.

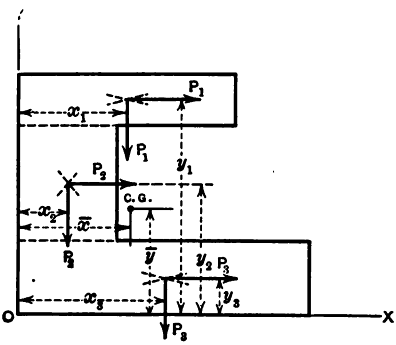


FIG. 25.

Algebraic Application. — Required the centre of gravity of the area of Fig. 25. Divide the area into rectangles as shown. The centre of gravity of each rectangle is evidently at the intersection of the diagonals. Apply at each centre thus obtained, a force in proportion to the area of the rectangle, giving forces P_1, P_2 and P_3. Choose as the axis of reference $O\,X$ and $O\,Y$ common to two sides of the area. The coordinates of each force are given by x and y with proper subscripts.

From eq. (2) we have

$$\bar{x} = \frac{P_1 x_1 + P_2 x_2 + P_3 x_3}{P_1 + P_2 + P_3}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\bar{y} = \frac{P_1 y_1 + P_2 y_2 + P_3 y_3}{P_1 + P_2 + P_3}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The centre of gravity (C. G.) is thus definitely located with reference to the coordinate axis.

Graphical Application.—Required the centre of gravity of the area of Fig. 26. Divide the area into rectangles as before. Find the resultants of the vertical and of the horizontal forces by the methods of Sec. III. The point of intersection of these resultants will give the centre of gravity of the area. The construction is shown in Fig. 26.

40. Moments of Forces.—*Graphical Method.*—The moment of a force has been defined in Art. 19, as its tendency to produce rotation about a point. It is measured by the product of the magnitude of the

force and the length of the perpendicular dropped from the point upon the line of action of the force. When the direction of the rotation is clockwise it will be called a positive moment. Counter clockwise moments will be called negative.

Given a system of forces, required the moment of the forces about a given point.

Let P_1–P_4, Fig. 27, be the given system of forces. Let P be the point about which the moment is required. Draw the force and

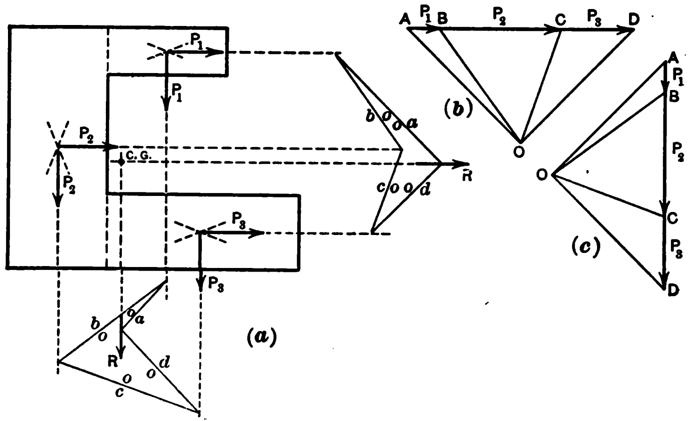


FIG. 26.

equilibrium polygons for the given system and determine the resultant R. Since the sum of the moments of all of the forces about point P is equal to the moment of the resultant about the same point, we have

$$M = P_1 r_1 + \ldots + P_4 r_4 = R r. \quad . \quad . \quad . \quad (5)$$

Draw through P a line parallel to R, intersecting the segments $o\,a$ and $o\,e$ at X and Z. In Fig. (*b*) drop the perpendicular $O\,F$ whose length $= H$. Then triangle $O\,A\,E$ is similar to triangle $Y\,X\,Z$, and we have $r : y = H : R$, or $R\,r = H\,y$; hence also

$$M = H\,y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The distance H is called the *pole distance* of the resultant R. H is measured in pounds to the scale of the force diagram, and y is measured in units of length to the scale of the space diagram.

In the same way the moment about the same point of only a part of the forces can be found. For example, required the moment of P_1 and P_2, of Fig. (*a*), about P. The resultant of P_1 and P_2 acts as shown in Fig. (*a*), and parallel to $A\,C$. Through P draw a line parallel to $A\,C$. The distance y_1, intercepted between the segments $o\,a$ and $o\,c$, adjacent to the forces P_1 and P_2, is the required intercept. The pole distance will be H_1, the perpendicular distance from O to $A\,C$. Then

$$M = H_1\,y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the given system of forces is in equilibrium, as is often the case, the value of R is zero and the moment of the entire system about any point is zero. The moment of a part of the forces is found as before.

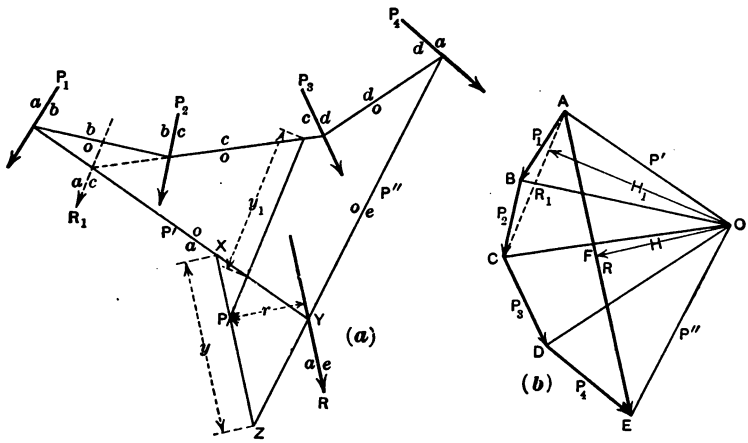


FIG. 27.

Therefore, in general, *the moment of any system of forces, about a point, is equal to the pole distance of the resultant of the forces, multiplied by the distance intercepted between adjacent segments of the equilibrium polygon on a line drawn through the given point parallel to the resultant.*

For parallel forces the method is the same as given above. The problem, which usually arises in practice is to determine the moment about a point of a part of a system of forces which is equilibrium, as, for example, the loads and reactions acting upon a beam or framed structure. To illustrate, let P_1–P_5, Fig. 28, be a system of forces in

equilibrium; required the moment of P_1 and P_2 about point P. The figure shows the equilibrium and force polygons already constructed. Draw a vertical through P giving the intercept y. Then, as in eq. (6), the desired moment $= H\,y$.

For parallel forces the pole distance, H, is constant, so that the equilibrium polygon becomes a moment diagram for moments of all forces to the left or right of any given point. The moments of any other combination of adjacent forces, taken about P, can be found by getting the intercept between the corresponding segments of the equilibrium polygon (produced if necessary). Thus the moment of forces P_2, P_3 and P_4 is equal to H times the intercept on the vertical through P measured between $o\,b$ and $o\,e$, produced.

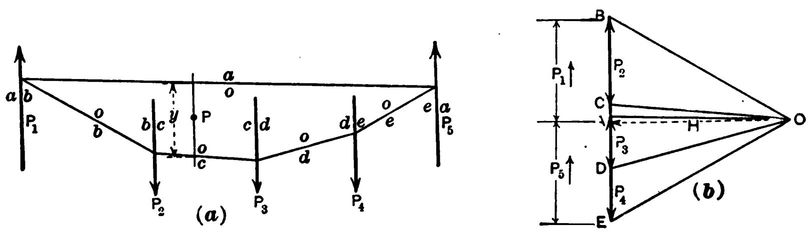


FIG. 28.

41. Moments of Inertia.—The moment of inertia of a force with respect to any axis is the product of its magnitude into the square of its distance from the axis. The total moment of inertia for any system of parallel forces will be the sum of such products for each force.

Thus

$$I = P_1\,a_1^2 + - - - - - - P_n\,a_n^2 = \Sigma\,P\,a^2. \quad . \quad . \quad (8)$$

Since $P\,a$ is the moment of the force in question about the given axis, eq. (8) can be written thus

$$I = M_1\,a_1 + - - - - - - - M_n\,a_n = \Sigma M\,a. \quad . \quad . \quad (9)$$

Eq. (9) furnishes a method by which moments of inertia may be calculated graphically. Consider the moment of the given force to be replaced by a force equal in magnitude to the amount of the moment acting in the line of action of the original force, and in the direction indicated by the sign of the moment. Multiply this new force by the

distance from the axis of moments. This will give the moment of inertia of the force or the "Second Moment" as it is sometimes called. The sum of all such second moments will give the total moment of

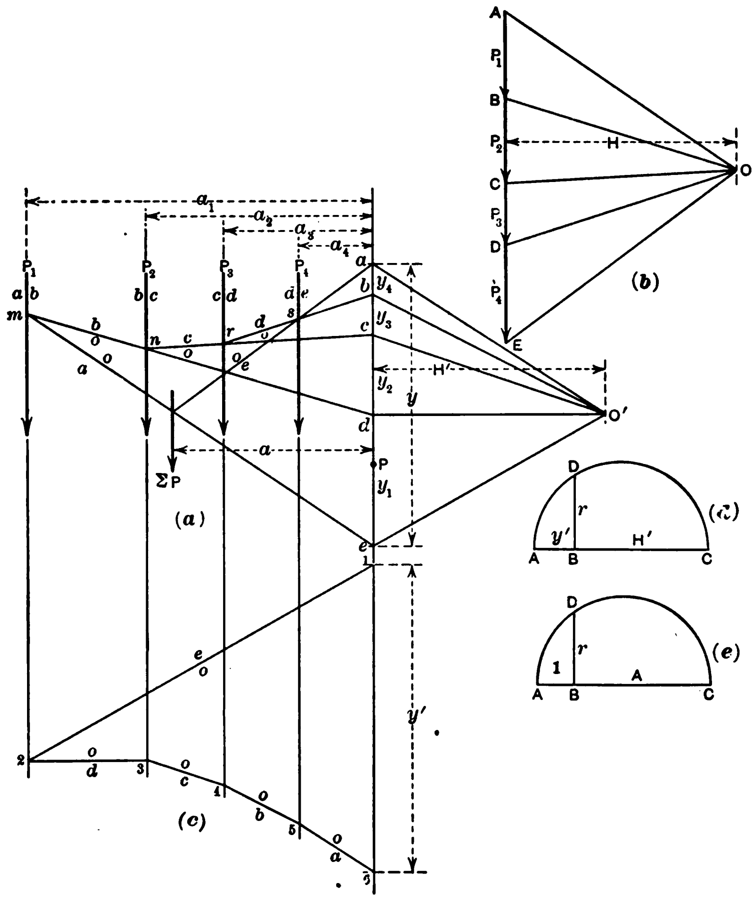


FIG. 29.

inertia for the system of forces. This method will be illustrated by the following problem:

Let P_1–P_4, Fig. 29, be a system of parallel forces, and P the point about which the moment of inertia is desired. Construct first the force and equilibrium polygons for these forces; these are shown in Figs.

(*a*) and (*b*). From the previous article, the moments of P_1, P_2, P_3 and P_4 about P are equal to $H\,y_1$, $H\,y_2$, $H\,y_3$ and $H\,y_4$ respectively. If these moments be applied as forces in the lines of action of the forces producing them, a new set of forces is obtained. Again take moments about P. Since H is constant, the new force polygon can be constructed by using y_1, y_2, y_3 and y_4 as the forces, and with O' as pole. Constructing the corresponding equilibrium polygon gives 1 2 3 4 5 6. The intercept now is y', and the second moment is $H'\,y'$. Since the second force polygon was constructed for the intercepts y_1–y_4 instead of the moments $H\,y_1$–$H\,y_4$, it is necessary to multiply the second moment $H'\,y'$ by H which gives $H\,H'\,y'$ as the required moment of inertia of the system.

It is possible to find the moment of inertia of the forces without constructing the second equilibrium polygon. This can be done as follows: We have as before

$$I = \Sigma\, P\, a^2 = \Sigma\, M\, a = \Sigma\, H\, y\, a.$$

This may be written, for the same case as before,

$$I = H\,(y_1\,a_1 + y_2\,a_2 + y_3\,a_3 + y_4\,a_4). \quad . \quad . \quad . \quad (10)$$

From Fig. 29, $y_1\,a_1$ is equal to twice the area of the triangle $m\,d\,e$, bounded by the intercept y_1 and the segments of the equilibrium polygon $o\,a$ and $o\,b$ for force P_1. In the same way $y_2\,a_2$, $y_3\,a_3$ and $y_4\,a_4$ represent similar areas. The sum of these areas is equal to twice the area of the figure enclosed by the equilibrium polygon and a vertical through P; it may be called the area of the equilibrium polygon. If this area is denoted by A we have

$$I = 2\,H\,A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

As the pole distance, H, can be chosen at will it may be taken equal to one-half the sum of the forces P_1–P_4, or $\Sigma\, P/2$. Eq. (11) then takes the convenient form

$$I = A\,\Sigma\, P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The moment of inertia of an area may be found by dividing the area into smaller areas. At the centre of gravity of each small area,

a force is applied which is in proportion to the area. Then by proceeding as before, the moment of inertia may be found. In such a case ΣP, in eq. (12), may be replaced by the area of the figure in question.

42. Radius of Gyration.—The radius of gyration of a system of forces is the distance from the axis at which a force equal to the resultant of the system must act in order to give the same moment of inertia as the forces themselves. That is,

$$I = r^2 \Sigma P \quad \text{or} \quad r^2 = \frac{I}{\Sigma P}. \qquad (13)$$

The radius of gyration, r, may be found from eq. (13) or graphical methods may be used directly. From the first method given in the preceding article we have

$$I = H H' y',$$

then

$$r^2 = \frac{I}{\Sigma P} = \frac{H H' y'}{\Sigma P}. \qquad (14)$$

If the pole distance, H, had been chosen equal to ΣP we would have

$$r^2 = \frac{\Sigma P \ H' y'}{\Sigma P} = H' y',$$

from which

$$r = (H' y')^{\frac{1}{2}}. \qquad (15)$$

This may be constructed graphically as shown in Fig. 29 (d). Lay off $A B = y'$ and $B C = H'$. On $A C$ as a diameter construct the semicircle $A D C$. From B erect a perpendicular $B D$. Then $B D^2 = A B \times B C$, hence $r = B D$.

The same result is obtained by the second method, eq. (12). Here $I = A \Sigma P$ where ΣP is the resultant of the system of forces and A is the area of the equilibrium polygon. As before $r^2 = \frac{I}{\Sigma P} = \frac{A \Sigma P}{\Sigma P} = A$. Here the diameter of the semicircle $A C$ will be made up of $A B = 1$, and $B C = A$. Then, as before, $r = B D$. Fig. 29 (e) shows the necessary construction.

SECTION V.—APPLICATION OF THE LAWS OF EQUILIBRIUM

43. General Methods.—The external forces acting on a structure are due to the weight of the structure itself, to the loads carried, and to the forces occurring at the points of support, called reactions. Of these external forces the loads are usually known while the reactions are unknown. The unknown reactions may be determined by the principles of equilibrium, since the forces acting on a structure at rest form a balanced system to which may be applied the three equations of Art. 29. We can therefore in general fully determine these external forces, provided there are not more than three unknowns.

Since all parts of a structure at rest are in equilibrium, we may evidently apply the laws of equilibrium to the forces acting upon any portion of that structure in order to obtain the stresses in the members. The portion considered may be a single joint, or it may include several joints and members. The forces acting upon the portion may be part external forces, part internal forces or stresses, or they may be wholly stresses. It may be said, therefore, that there are two general methods of applying the equations of equilibrium to framed structures:

1. To the structure as a whole to determine reactions, and

2. To a part of the structure to determine stresses. Under (2) we have:

(*a*) Application to any single joint, the forces being concurrent. This method is called the *method of successive joints.*

(*b*) Application to a portion of the structure including several joints, which are considered as separated from the remaining part by passing a section through the structure. The forces in this case are generally non-concurrent. This method is called the *method of sections.*

In the following articles a few examples will be given to illustrate the application of the laws of equilibrium in finding reactions and stresses by both algebraic and graphical methods.

44. First.—To the Structure as a Whole to Determine Reactions.—*Algebraic Method.*—Example 1. Suppose the roof truss of Fig. 30 to be acted upon by the wind pressure W, acting normally to the roof; the weight, G, of the roof and truss applied at their centre of gravity and acting downward; and the abutment reactions, as yet unknown in

amount or direction. These comprise all the external forces. Fig. (*b*) shows the truss free from the abutments, with the reactions R' and R'' indicated. The left end of the truss is supposed to rest on rollers, the abutment at B taking all the horizontal thrust due to wind. This being

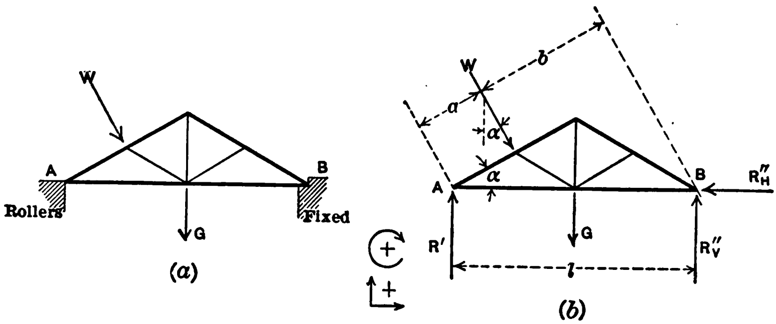


FIG. 30.

the case, R' will be vertical and R'' inclined at an angle θ with the vertical. Let R″ be replaced by its vertical and horizontal components. The unknowns are R', R''_H, R''_V.

Applying the three equations of equilibrium, choosing two moment and one resolution equations gives:

From $\Sigma H = 0$

$$W \sin \alpha - R''_H = 0,$$

from which

$$R''_H = W \sin \alpha. \quad \ldots \ldots \quad (a)$$

From $\Sigma M_A = 0$

$$+ W a + \frac{G l}{2} - R''_V l = 0,$$

from which

$$R''_V = + \frac{W a}{l} + \frac{G}{2}. \quad \ldots \ldots \quad (b)$$

From $\Sigma M_B = 0$

$$- W b - \frac{G l}{2} + R' l = 0,$$

from which

$$R' = \frac{W b}{l} + \frac{G}{2}. \quad \ldots \ldots \quad (c)$$

From (*a*) and (*b*) we have

$$R'' = [(R''_H)^2 + (R''_V)^2]^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (d)$$

also

$$\theta = \tan^{-1} \frac{R''_H}{R''_V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

The unknowns are therefore completely determined.

Example 2. Given the bridge truss of Fig. 31; loads as shown; rollers at A. Required the abutment reactions. Fig. (*b*) shows the truss free of the abutments. Since there are rollers at A, R' will be

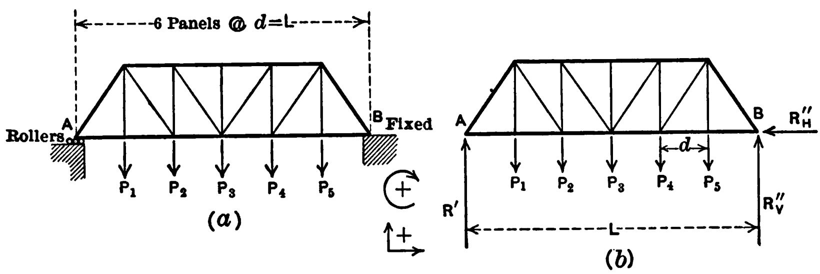


FIG. 31.

vertical. R'' will have vertical and horizontal components, since any horizontal force must be resisted here. Our unknowns are thus R', R''_H and R''_V.

Applying the three equilibrium equations to Fig. (*b*) we have:

From $\Sigma H = 0$

$$-R''_H = 0,$$

from which

$$R''_H = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

From $\Sigma M_A = 0$

$$-R''_V L + P_1 d + P_2 2d + P_3 3d + P_4 4d + P_5 5d = 0,$$

from which

$$R''_V = \frac{(P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5)d}{L}. \quad . \quad . \quad . \quad (b)$$

From $\Sigma M_B = 0$

$$+ R' L - P_5 d - P_4 2 d - P_3 3 d - P_2 4 d - P_1 5 d = 0$$

from which

$$R' = \frac{(P_5 + 2 P_4 + 3 P_3 + 4 P_2 + 5 P_1) d}{L} \quad . \quad . \quad . \quad . \quad (c)$$

The fact that $R''_H = 0$ as given in equation (*a*) might have readily been determined by inspection, there being no other horizontal force. In arriving at conclusions by inspection we must, however, be very careful to see that they are based upon some one of the three conditions of equilibrium, and where the result is doubtful we should always return to the rigid method: Consider the structure by itself, put in all forces, and write out in detail the equations of equilibrium.

Example 3. Given the bridge truss of Fig. 32; loads as shown; rollers at *A*. Required the abutment reactions.

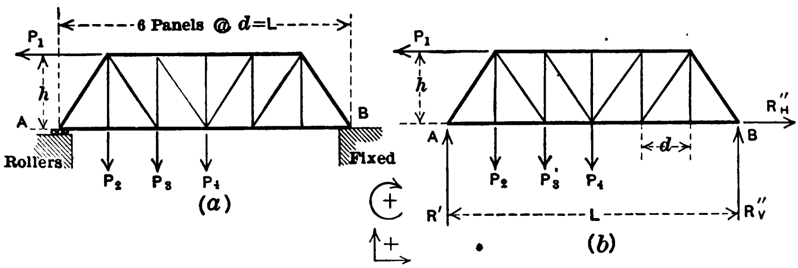


FIG. 32.

Fig. (*b*) shows the structure removed from the abutments and the unknown reactions R', R''_V and R''_H indicated in position. Proceeding as in Example 2, we have:

From $\Sigma H = 0$

$$+ R''_H - P_1 = 0,$$

from which

$$R''_H = + P_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

From $\Sigma M_A = 0$

$$- P_1 h + P_2 d + P_3 2 d + P_4 3 d - R''_V L = 0,$$

from which

$$R''_V = \frac{- P_1 h + (P_2 + 2 P_3 + 3 P_4) d}{L} \quad . \quad . \quad . \quad . \quad (b)$$

From $\Sigma M_B = 0$

$$-P_1 h - P_4 3d - P_3 4d - P_2 5d + R' L = 0,$$

from which

$$R' = \frac{+P_1 h + (3P_4 + 4P_3 + 5P_2)d}{L}. \quad . \quad . \quad . \quad (c)$$

The unknowns are thus completely determined.

Example 4. Given the three hinged arch of Fig. 33; hinges at A, B, and C; loads as shown. Required reactions at the hinges.

Consider first the structure as a whole removed from the abutments, and the unknown reaction R'_H, R'_V, R''_H, R''_V placed in position. As before, moment equations about A and C will give the values of R'_V and R''_V respectively.

From $\Sigma M_A = 0$

$$-R''_V (l_1 + l_2) + P_1 b_1 + P_2 (l_1 + a_2) = 0,$$

from which

$$R''_V = \frac{P_1 b_1 + P_2 (l_1 + a_2)}{(l_1 + l_2)}. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

From $\Sigma M_C = 0$

$$+R'_V (l_1 + l_2) - P_1 (l_2 + a_1) - P_2 b_2 = 0,$$

from which

$$R'_V = \frac{P_1 (l_2 + a_1) + P_2 b_2}{(l_1 + l_2)}. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

A resolution equation involving horizontal forces gives

$\Sigma H = 0$

$$R'_H - R''_H = 0,$$

from which

$$R'_H = R''_H. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Eq. (c) gives only an equality between R'_H, and R''_H. In order to obtain the value of R'_H and R''_H, we must have another independent equation.

Since the structure is hinged at B it can be separated at that point and the action of one part on the other will be fully represented by a single inclined force, or the two components of such a force. Figs. (c) and (d) show the structure separated and the external forces placed in

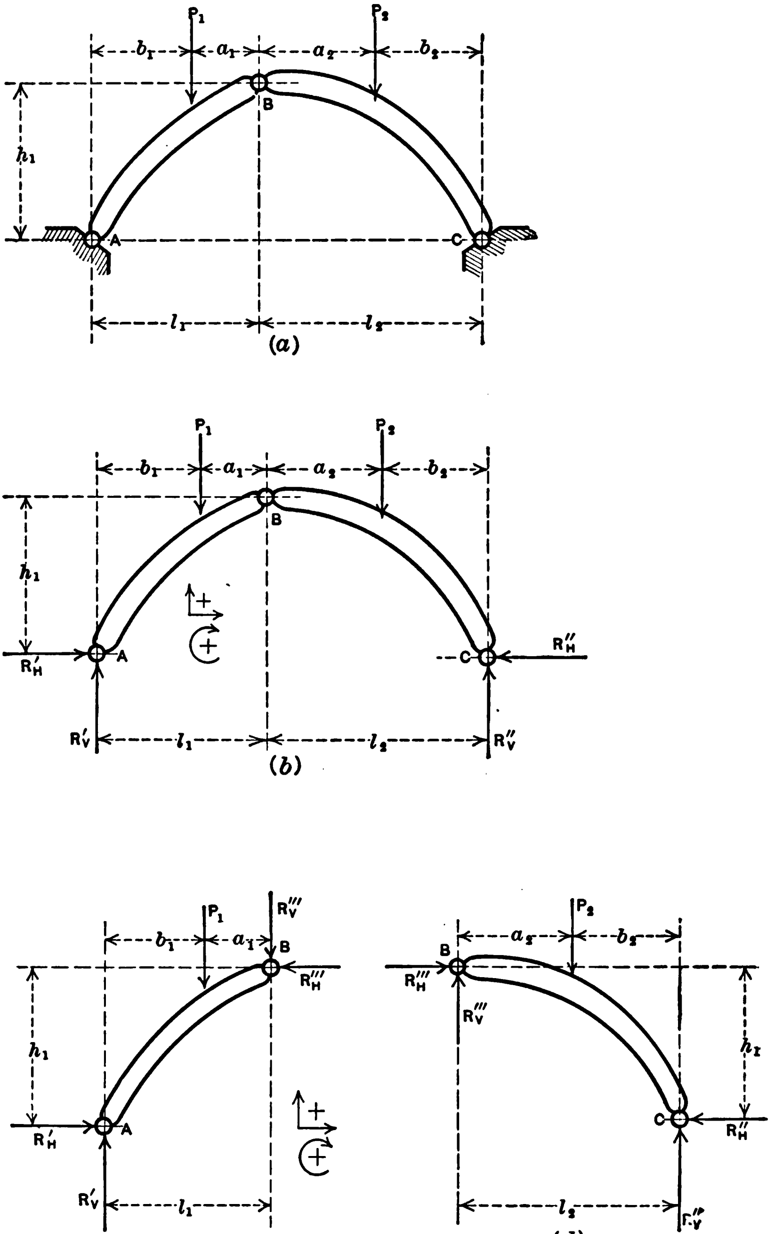


FIG. 33.

position, the horizontal and vertical components of the forces being indicated. At B we have a simple case of action and reaction; the forces acting upon the two parts are equal and opposite. Since the values of R'_V and R''_V are known from eqs. (a) and (b) above, we have now only three unknowns for each part of the structure—R'_H, R'''_H and R'''_V, for the left half, and R''_H, R'''_H and R'''_V for the right half. Applying the three equilibrium equations to the left half, we have:

From $\Sigma V = 0$

$$+ R'_V - R'''_V - P_1 = 0,$$

from which

$$R'''_V = R'_V - P_1. \qquad (d)$$

From $\Sigma M_A = 0$

$$+ R'''_V l_1 - R'''_H h_1 + P_1 b_1 = 0,$$

from which

$$R'''_H = \frac{R'''_V l_1 + P_2 b_1}{h_1}. \qquad (e)$$

From $\Sigma H = 0$

$$+ R'_H - R'''_H = 0,$$

from which

$$R'_H = R'''_H. \qquad (f)$$

In the same way for the right half, we have:

From $\Sigma V = 0$

$$+ R'''_V + R''_V - P_2 = 0,$$

from which

$$R'''_V = + P_2 - R''_V. \qquad (g)$$

From $\Sigma H = 0$

$$+ R''_H - R'''_H = 0,$$

from which

$$R''_H = + R'''_H. \qquad (h)$$

From $\Sigma M_C = 0$

$$+ R'''_H h_1 - P_2 b_2 + R'''_V l_2 = 0,$$

from which

$$R'''_H = \frac{+ P_2 b_2 - R'''_V l_2}{h_1}. \qquad (i)$$

Example 5. Cantilever Bridge, Fig. 34. Links at A and F; hinges at B, D, and E, and rollers at C; loads as shown; weight of structure neglected. Find reactions at A, B, E, and F.

The structure may be divided into three parts as shown in Figs. (*b*), (*c*), and (*d*). The reactions at the various points are shown on the figures. Since links are placed at A and F and rollers at C the reactions at these points must be vertical. At B, D, and E, which are hinged only, horizontal components must be considered. The three equi-

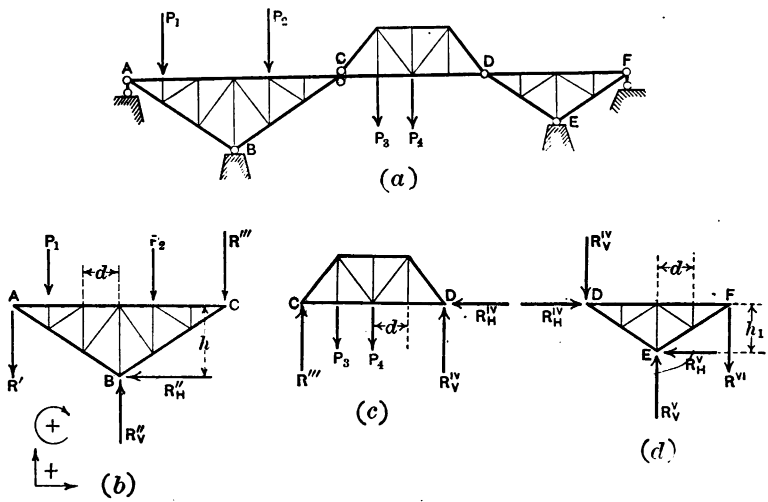


FIG. 34.

librium equations may then be applied to these three structures. Beginning with the center structure, the unknowns R''', R^{IV}_H and R^{IV}_V may be determined as in Example 2. These values may then be applied as known forces in their proper places in the side spans and the remainder of the unknowns determined.

45. *Graphical Method.*—Example 1. Given the roof truss of Fig. 35, acted upon by the wind load W, normal to the roof, the weight G, of the truss and roof acting vertically, and the unknown reactions R_1 and R_2. The left end of the truss is supposed to be on rollers, hence the reaction R_1 will be vertical. R_2 is known only in point of application. The unknown can be determined by the method of Art. 44. Beginning with the known forces W and G, draw the corresponding portion of the

force polygon $B\,C\,D$ of Fig. (*b*). Choose a convenient pole O and draw the rays $O\,B$, $O\,C$ and $O\,D$. Draw the portion of the equilibrium polygon, 1–2–3–4, corresponding to these rays. Since point 1 is known to be on the line of action of R_2, the equilibrium polygon must be started here. Draw the closing line 1–4. In the force polygon draw from point O the ray $O\,A$ corresponding to the closing line. The point

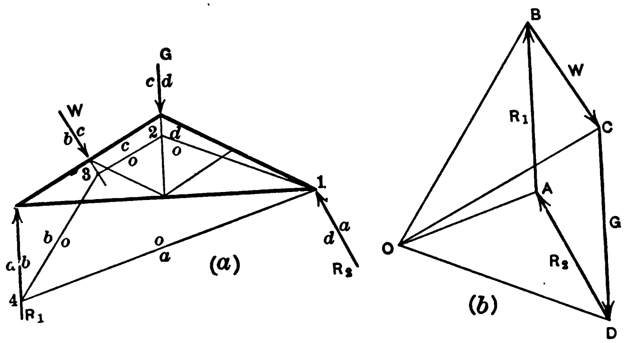


FIG. 35.

of intersection, A, of $O\,A$ with a line drawn from B parallel to the known direction of R_1 will give the amount of the left reaction. Thus R_1 is represented in magnitude by the line $A\,B$. R_2 is given in amount and direction by $D\,A$. R_1 and R_2 are then completely determined.

Example 2. Given the roof truss of Fig. 36 with loads P_1–P_5 and wind pressures W_1–W_3. Required the abutment reactions R_1 and R_2; (*a*), when a part of the horizontal thrust due to wind is taken up by each abutment, the truss being fixed at each end; (*b*), when one end is on rollers, and the horizontal component of the pressure is all resisted at the other end.

Construct the force polygon, Fig. (*b*), laying off the forces in any convenient order. The order chosen here is P_1–P_5, W_1–W_3. Consider first case (*a*). Both ends being fixed the exact direction of the reactions is indeterminate, but it will answer present purposes to assume them parallel and having a direction parallel to the resultant load. (Another assumption often made is that their horizontal components are equal.) Choose any pole O, and draw the rays $O\,A$–$O\,K$. Draw the equilibrium polygon 1, 2, 3,–10, starting at any point 1 on the line of action R_1 and ending at 10, a point on R_2; 1–10 is the closing line.

Draw the corresponding ray $O\,L$. $K\,L$ and $L\,A$ are then respectively equal to the required reactions R_1 and R_2 acting as shown by the arrows.

Case (*b*) is similar to Example 1. The construction is shown by dotted lines. The right end is assumed to be on rollers, thus giving

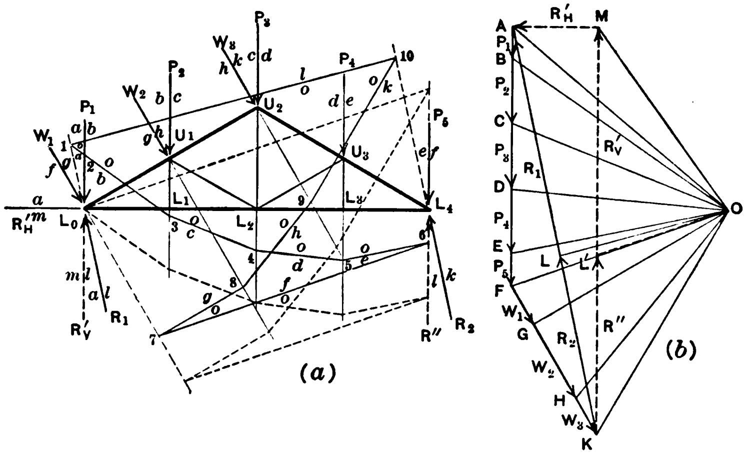


FIG. 36.

a vertical reaction. The equilibrium polygon should be started at L_o, the left abutment, as this is the only point known on the line of action of R'.

Example 3. Given the bridge truss of Fig. 37, acted upon by the vertical loads P_1–P_5. Required the reactions R_1 and R_2.

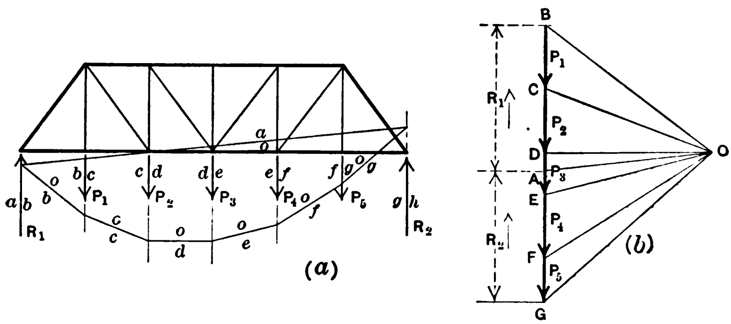


FIG. 37.

This is a case of vertical loads as in Art. 36. The construction is shown in Figs. 37 (*a*) and (*b*).

Example 4. Given the bridge truss of Fig. 38, acted upon by vertical loads P_2, P_3 and P_4 and by a horizontal load P_1. The truss is assumed as fixed at the left end. This is similar to Example 1, above. The construction is shown in Figs. 38 (*a*) and (*b*).

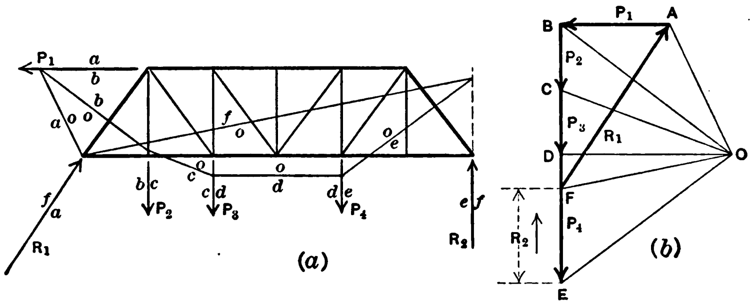


FIG. 38.

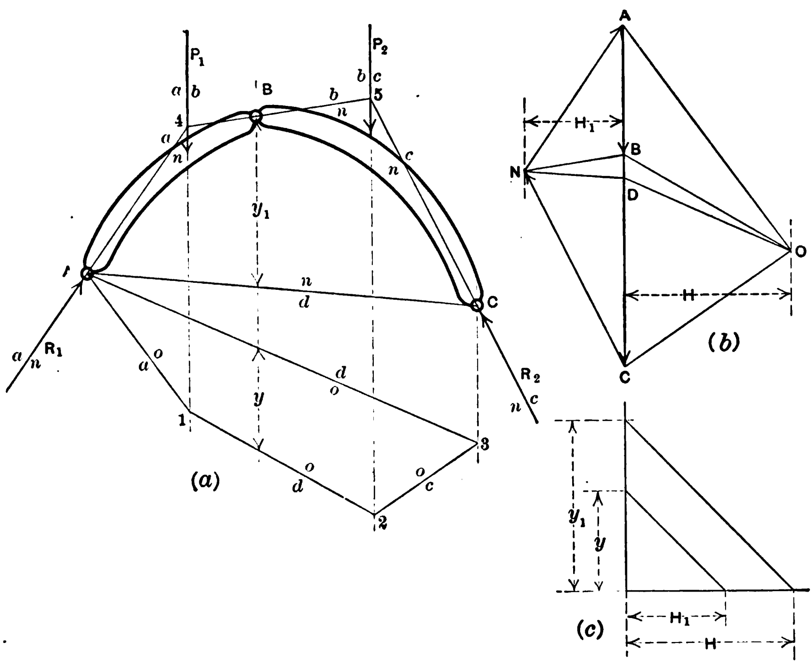


FIG. 39.

Example 5. Given the three hinged arch of Fig. 39; hinges at *A*, *B*, and *C*; loads as shown. Required the abutment reactions at

A and C. Here four unknowns are in question, the amounts and directions of two forces R_1 and R_2, which are known only in points of application, the hinges A and C.

The closure of the force and equilibrium polygons permits the determination of three unknowns. To determine the fourth we can make use of the fact that the hinges at A, B and C can resist only forces which pass through their centres. As the lines of action of these forces constitute the corresponding segments of the equilibrium polygon, it follows that the equilibrium polygon of the given forces must pass through the hinges. Therefore, as a fourth condition the equilibrium polygon must pass through the points A, B and C. Proceed, as in Art. 37, to pass an equilibrium polygon for the forces P_1 and P_2 through the required points. A-1-2-3 shows the trial polygon, pole chosen at O; and A-4-B-5-C shows the required equilibrium polygon with true pole found to be at N. The amount and direction of R_1 and R_2 are given respectively by $N A$, acting from N to A, and $C N$, acting from C to N.

46. Second. —To Single Joints to find Stresses.—*Algebraic Method.* This method consists in taking single joints at which not more than two unknowns exist. We then have a system of concurrent forces which can be solved by means of the principles of Art. 23.

Example 1. Given the roof truss of Fig. 40; loads as shown; required the stresses in all members.

First find the abutment reactions at A and B, treating the structure as a whole.

From $\Sigma M_B = 0$

$$+ R_1 l - \frac{W}{2} l - W \frac{3}{4} l - W \frac{l}{2} - W \frac{l}{4} = 0,$$

from which

$$R_1 = + 2 W.$$

Since the loads are symmetrical about the centre, the reaction at B is equal to that at A; hence $R_2 = 2 W$. As there are but two members meeting at joint A the determination of the stresses in the members will be started at this point. Fig. (b) shows the joint at A removed and the known forces applied, together with the unknowns S_1 and

S_2, assumed to act as shown. Applying the two equations of equilibrium to the joint we have:

From $\Sigma V = 0$

$$+ 2W - \frac{W}{2} - S_1 \sin\theta = 0,$$

from which

$$S_1 = + \frac{3}{2} W \operatorname{cosec}\theta. \quad \dots\dots\dots \quad (a)$$

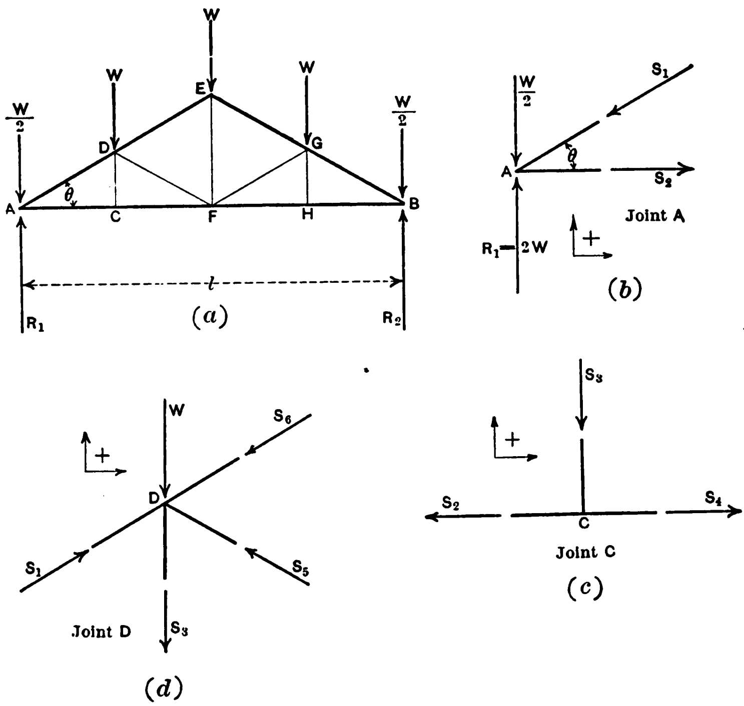


FIG. 40.

From $\Sigma H = 0$

$$+ S_2 - S_1 \cos\theta = 0$$

from which

$$S_2 = + S_1 \cos\theta$$

or

$$S_2 = \frac{3}{2} W \operatorname{cosec}\theta \cos\theta = \frac{3}{2} W \cot\theta, \quad \dots \quad (b)$$

The sign of the results will show whether the correct directions were assumed.

Passing now to the next joint at which only two unknowns exist, we select joint C, shown in Fig. (c). Here S_2 is known from the calculations for joint A and is given in eq. (b). S_3 and S_4 are the unknowns, assumed to act as shown. Applying the equations of equilibrium gives

From $\Sigma H = 0$

$$+ S_4 - S_2 = 0,$$

from which

$$S_4 = + S_2 = + \frac{3}{2} W \cot \theta. \quad \text{(c)}$$

From $\Sigma V = 0$

$$S_3 = 0. \quad \text{(d)}$$

Next pass to joint D, which is shown in Fig. (d). S_1, S_3 and W are known forces. The two unknowns are S_5 and S_6. Applying the two equations of equilibrium gives

From $\Sigma H = 0$

$$+ S_1 \cos \theta - S_6 \cos \theta - S_5 \cos \theta = 0.$$

From $\Sigma V = 0$

$$+ S_1 \sin \theta - S_6 \sin \theta + S_5 \sin \theta - W = 0.$$

We have here two independent equations involving the unknowns S_5 and S_6. Solving these simultaneously gives

$$S_6 = + W \operatorname{cosec} \theta. \quad \text{(e)}$$

$$S_5 = + \frac{W}{2} \operatorname{cosec} \theta. \quad \text{(f)}$$

The stresses at D are thus completely determined. In the same way pass to the other joints, selecting in each case a joint at which but two unknowns exist.

Example 2. Bridge truss of Fig. 41; loads as shown. Determine first the reactions R_1 and R_2. Then, as in Example 1, start at joint A,

taking the joints in succession in the order shown in the figure. In this problem it will be found in each case that, by writing the equations in the proper order, only one unknown will appear at a time, thus making the solution simple.

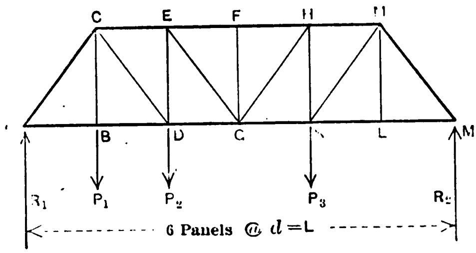


FIG. 41.

Example 3. Roof truss of Fig. 42; loads as shown. At first sight this truss appears to be indeterminate, due to the fact that three unknowns exist at joint A. However, by taking first joint B, resolving forces parallel and perpendicular to member $A\ B\ D$, the stress in $B\ C$ can be found. Then pass to joint C and determine the stress in $A\ C$. The stress in $A\ C$, once known, reduces the unknowns at joint A to two, allowing a solution for stresses in $A\ B$ and $A\ G$. The methods employed are the same as used in Example 1, above.

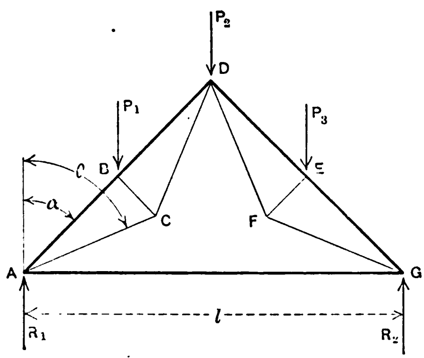


FIG. 42.

47. *Graphical Method.*—As in the algebraic method, joints are taken at which only two unknowns exist. The system is then solved graphically as explained in Art. 30.

Example 1. Roof truss of Fig. 43; loads as shown. Required the stresses in all members. First determine the reactions R_1 and R_2 by the methods of Art. 35. Then as joint L_0 is the only one at which we

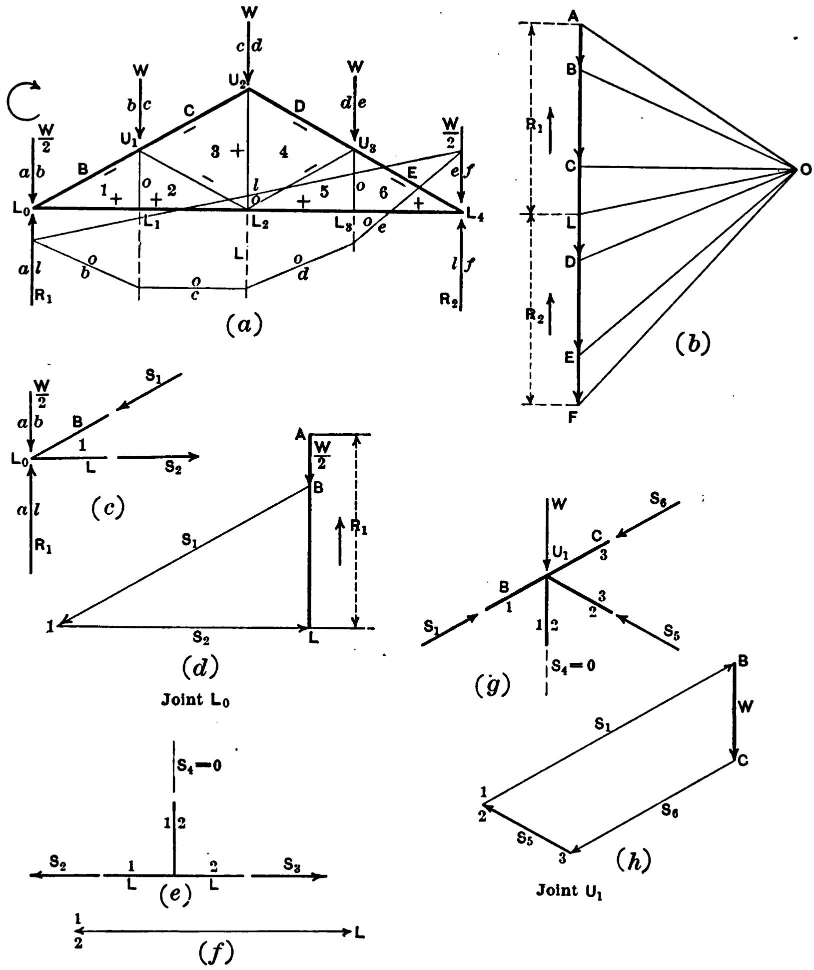


FIG. 43 (*a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*).

have two unknowns we must start there. Fig. (*c*) shows this joint removed, and all the forces applied. The unknowns are represented by S_1 and S_2. First lay off the known forces, $\frac{W}{2}$ acting downward and

R_1 acting upward. From B draw a line parallel to the line of action of S_1 and from L draw a line parallel to the line of action of S_2. The point of intersection, 1, of these two lines will determine the magnitudes of S_1 and S_2. The direction in which these forces act can be found by passing around the force polygon in the direction of the arrows. S_1 is found to act toward the joint, indicating compression, while S_2 acts away from the joint, indicating tension.

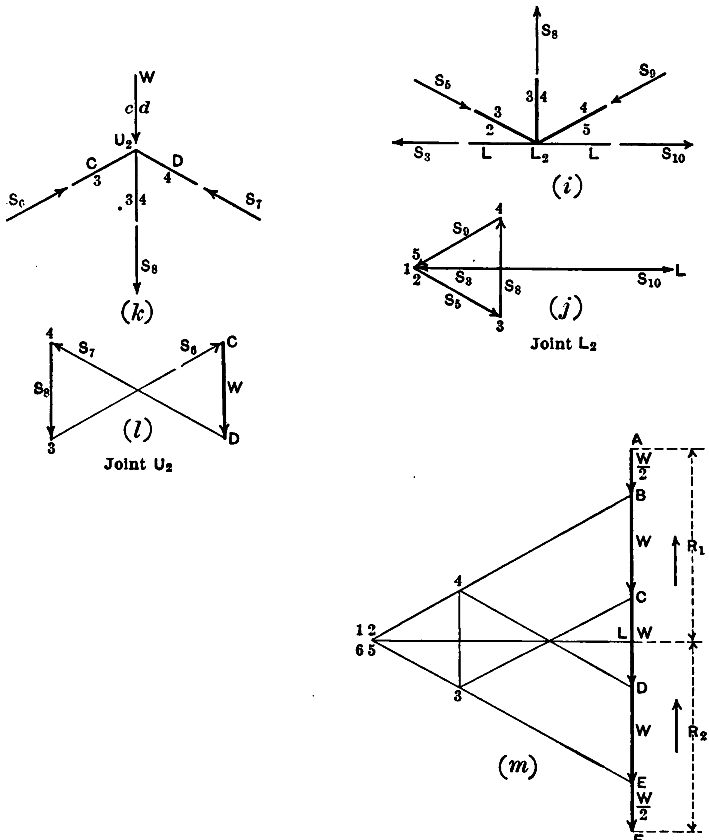


FIG. 43 (*i*, *j*, *k*, *l*, and *m*).

The joint at L_1 is the next one at which only two unknowns exist. Fig. (*e*) shows this joint with the forces indicated. The stress S_2 is known from joint L_0, and S_3 and S_4 are unknown. Lay off S_2, Fig. (*f*), as determined from L_0 on a line parallel to the member, with the

force acting from L to 1. We must now close the force polygon with lines parallel to S_3 and S_4. But since S_3 has the same line of action as S_2, the line in the force diagram representing the magnitude of S_4 will be a point, thus having no length. The stress in S_4 is therefore zero. This might have been seen by inspection, as there is no load at L_1 acting so as to cause stress in S_4.

Next pass to joint U_1 shown in Fig. (*g*). Here S_1 and S_4 are known from joints L_0 and L_1. In the same way pass to joints U_2, L_2, L_3, U_3 and L_4 in the order named. The force diagrams for joints L_2 and U_2 are given in Figs. (*i*), (*j*), (*k*), and (*l*). As the stresses in the two halves of the truss are symmetrical the stress diagrams in the right half are not given. When the loads are not symmetrical, the diagrams would have been drawn for all joints.

Instead of drawing a separate figure for each joint, we may combine the force polygons, using each line twice as being common to two polygons. The combined figure may be called a *stress diagram.* A convenient notation to accompany this method is to letter each triangle of the truss, and also each space between the external forces, as in Fig. (*a*). Each piece and each external force is then known by the two letters in the adjacent spaces, as the piece B 1 and the load $B C$, etc. Let us apply this method to the truss in Fig. 43 (*a*). Lay off the loads and determine the reactions as in Fig. (*b*). Beginning as before, at joint L_0, the force polygon for that joint will be $L A B$ 1 L, Fig. (*m*); B 1 is the stress in B 1 and L 1 is the stress in L 1. In treating a joint the forces are always taken in the order in which the loads were plotted in the load-line. In this case the loads were laid off from left to right; that is, in a clockwise direction. Therefore, we must pass around each joint in the same direction, beginning with the piece farthest to the left whose stress is known. At joint L_0 the order will be R_1, $\frac{W}{2}$, then piece B 1 and finally piece L 1. The circular arrow at the left of Fig. (*a*) is intended to indicate the direction to be used, which should be noted at the time the load-line is laid off.

The next step is to determine the nature of the stresses in the members. This can be done by following around the polygon. Beginning as before with R_1, Fig. (*m*), we pass from L to A, since R_1 acts

upward. $\frac{W}{2}$ acts downward, from A to B. Continuing around the polygon, B 1 acts toward joint L_0, indicating compression, and L 1 acts away from L_0, indicating tension. For future reference, mark on the member the kind of stress existing. Thus in Fig. (*a*) the − and + signs indicate respectively compression and tension.

Passing to joint L_1, the known stress farthest to the left is the tension in L 1, acting from L to 1. The next member in order is 1 2 and then L 2. To complete the polygon we must close on point L with a line

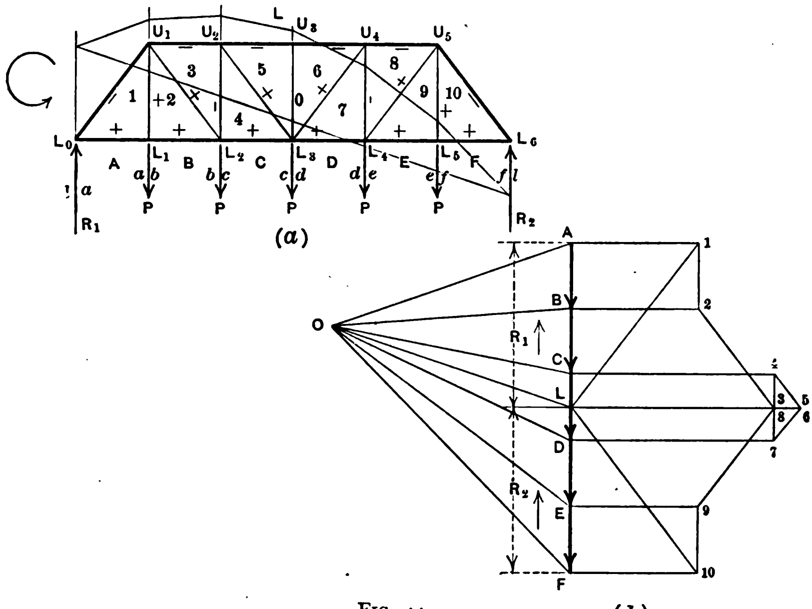


FIG. 44.

parallel to member L 2. To accomplish this the line representing stress in member 1 2 can have no length, therefore indicating zero stress. This fact is shown in the force polygon by placing letters 1 and 2 at the same point. Next pass to joint U_1. The force polygon will be 1 B C 3 2 1. The force polygon for joint L_4 is L 6 E F L, F L being the abutment reaction, which indicates a check on the work.

Example 2. Bridge truss of Fig. 44; loads as shown. Required the stresses in all members. First find the abutment reactions R_1 and

R_2. The graphical solution is shown in the figure. The stress diagram is given in Fig. (b). The diagram is begun at joint L_0, the remaining joints being taken up in the following order: L_1, U_1, L_2, U_2, U_3, L_3, etc. The order in which the forces are taken around the joint is shown by the arrow to the left of the figure and the kind of stress is indicated on the truss diagram.

48. Third.—The Method of Sections.—*Algebraic Method.*—In this method, instead of taking single joints, a section is passed through the structure cutting the members whose stresses are desired, and the equations of equilibrium applied to one of the portions into which the structure is thus divided. A part of the forces are thus external forces, and a part are due to stresses in the members cut. The portion of the structure considered usually includes several joints, and thus the forces are in general non-concurrent. This gives us three equations of equilibrium, and hence if we cut but three members (not meeting in a point) whose stresses are unknown, these stresses may be found. Another way of stating the equilibrium existing between the forces acting upon either portion of the structure, is to say that the stresses in any section hold in equilibrium the external forces acting upon either side of that section. Or, more in detail,

Σ vert. comp. external forces $=$ Σ vert. comp. internal forces.

Σ hor. comp. external forces $=$ Σ hor. comp. internal forces.

Σ mom. external forces $=$ Σ mom. internal forces.

The equality is, however, one of numerical value but not of sign, as is seen by comparison with the fundamental equations of equilibrium.

Example 1. Roof truss of Fig. 45; loads as shown. Required the stresses in all members.

The reactions are first found. To find the stresses in members L_0U_1 and L_0L_1 pass a section a–b cutting these members. We then have a case of concurrent forces same as joint A of Fig. 40. As the solution is identical it will not be repeated here.

To determine the stresses in members U_1U_2, U_1L_2 and L_1L_2, pass a section c–d cutting these members and remove the portion of the structure to the left together with all external forces. Replace the stresses in the members by forces acting along the centre lines of these members, assuming their directions at will. Fig. 45 (b) shows the

resulting system of forces; S_1, S_2, and S_3 are forces representing the unknown stresses. We have here a system of non-concurrent forces

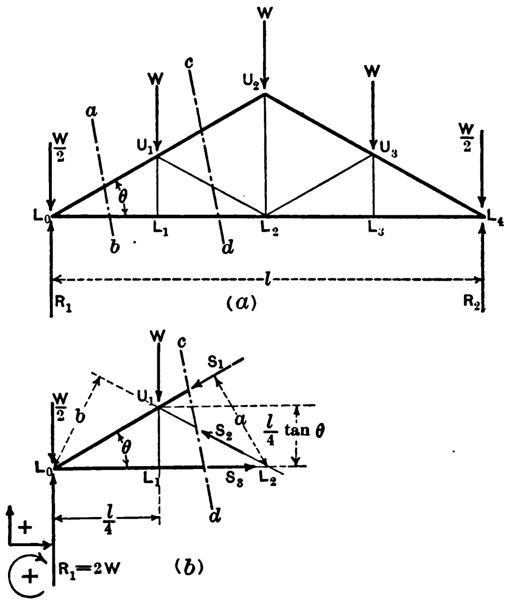


FIG. 45.

in equilibrium. Applying the three equations of equilibrium, using one moment and two resolution equations, we have

From $\Sigma V = 0$

$$- S_1 \sin\theta + S_2 \sin\theta - W - \frac{W}{2} + 2W = 0,$$

from which

$$- S_1 \sin\theta + S_2 \sin\theta + \frac{W}{2} = 0. \quad \text{. . . . .} \quad (a)$$

From $\Sigma H = 0$

$$- S_1 \cos\theta - S_2 \cos\theta + S_3 = 0. \quad \text{. . . . .} \quad (b)$$

From ΣM about $U_1 = 0$

$$+ 2W\frac{l}{4} - \frac{W}{2}\frac{l}{4} - S_3\frac{l}{4}\tan\theta = 0,$$

from which

$$S_3 = + \frac{3}{2} W \cot\theta. \quad \text{. . . . . . .} \quad (c)$$

The moment equation was taken about point U_1 in order to eliminate S_1 and S_2. Eq. (*c*) thus gives directly the value of S_3, which is found to be the same as for the corresponding member of Art. 46. Eqs. (*a*) and (*b*) can be solved simultaneously to get the values of S_1 and S_2. Substituting the value of S_3 in eq. (*b*) and solving gives

$$\left.\begin{aligned} S_1 &= + W \operatorname{cosec} \theta. \\ S_2 &= + \frac{W}{2} \operatorname{cosec} \theta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

The plus sign shows that the correct direction was assumed for the forces. By comparison with Art. 46, these values are found to check those obtained by the other method.

In some cases, a little study of the figure will show that the solution may be obtained in a simpler manner than the one here used. In the case of Fig. (*b*) the moment and resolution equations can be so taken that only one unknown will be contained in each equation, thus shortening the work.

Thus, to determine the stress in $U_1 U_2$, take moments about L_2, the intersection of $U_1 L_2$ and $L_1 L_2$, thus eliminating S_1 and S_3 which pass through the moment centre. The moment equation becomes

$$- S_1 a - W \frac{l}{4} - \frac{W}{2} \frac{l}{2} + 2 W \frac{l}{2} = 0,$$

from which

$$S_1 = + \frac{W l}{2 a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

but a, the level arm of S_1, is equal to $\frac{l}{2} \sin \theta$, which, substituted in eq. (*e*) gives

$$S_1 = + W \operatorname{cosec} \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

The stress in $U_1 L_2$ can be obtained by taking moments about L_0, the intersection of $U_1 U_2$ and $L_1 L_2$. The resulting moment equation is

$$- S_2 b + W \frac{l}{4} = 0,$$

from which

$$S_2 = + \frac{W l}{4 b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

The lever arm, b, is equal to $\frac{l}{2} \sin \theta$. Substituting the value of b in eq. (g) gives

$$S_2 = + \frac{W}{2} \operatorname{cosec} \theta. \qquad (h)$$

Example 2. Given the bridge truss of Fig. 46 (same as Fig. 44); loads as shown. Determine the stresses in members $U_1 U_2$, $U_1 L_2$ and $L_1 L_2$.

Pass a section a–b cutting the members whose stresses are desired. Fig. (b) shows the portion to the left of the section, with all forces indicated, together with the reactions as determined by the methods of

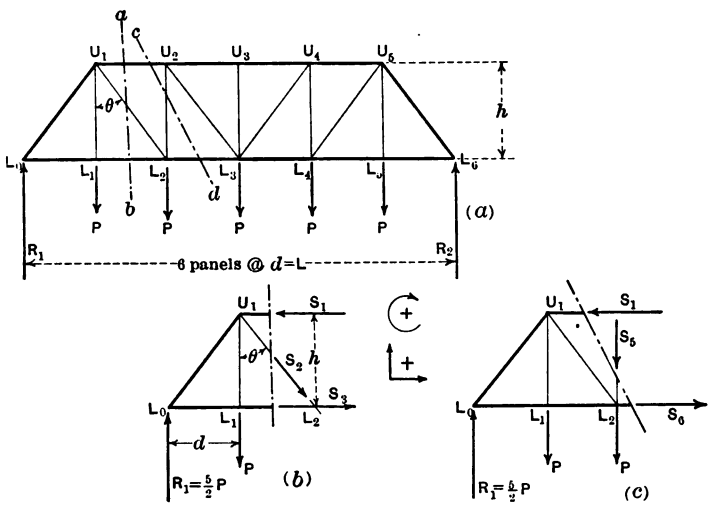


FIG. 46.

Art. 45. S_1, S_2, and S_3 are forces replacing the unknown stresses and are assumed to act as shown by the arrows. The value of S_2 can be determined by a resolution equation,

From $\Sigma V = 0$

$$- S_2 \cos \theta - P + \frac{5}{2} P = 0,$$

from which

$$S_2 = \frac{3}{2} P \sec \theta. \qquad (a)$$

The stress in S_3 can be determined by a moment equation about U_1 which gives

$$-S_3 h + \frac{5}{2} P d = 0,$$

from which

$$S_3 = +\frac{5}{2}\frac{P d}{h}. \qquad (b)$$

In the same way moments about L_2 gives the stress in S_1 thus

$$-S_1 h - P d + \frac{5}{2} P \times 2 d = 0,$$

from which

$$S_1 = +\frac{4 P d}{h}. \qquad (c)$$

To determine the stress in a vertical member, such as $U_2 L_2$, pass a section *c–d*, cutting the member in question and not more than two others. The stress in S_5 can be obtained by a resolution equation, putting $\Sigma V = 0$, which gives

$$-S_5 - P - P + \frac{5}{2} P = 0,$$

from which

$$S_5 = +\frac{P}{2}. \qquad (d)$$

Since S_1 and S_6 are both perpendicular to the axis of resolution, they have no vertical components and do not enter the equation.

From the above examples it can be seen that the method of sections has many advantages over the method of successive joints, for the former method allows a single member to be taken from any part of the structure and its stress determined without any reference to the other members of the truss, except those cut by the same section. The method of successive joints requires the determination of the stresses in all the truss members between the abutment and the desired member, thus making the work very long in the case of the members near the centre of the truss.

49. *Graphical Method.*—To apply this method we must commence at one end of the structure and pass a section cutting but two members,

whose stresses can be determined by the single condition that the force polygon, drawn for the forces on one portion of the structure, must close. Then passing another section cutting three members, one of which has already been treated, we can find the stresses in the other two, and finally, by successive sections taken in the same manner, we can determine all the stresses by simple force polygons.

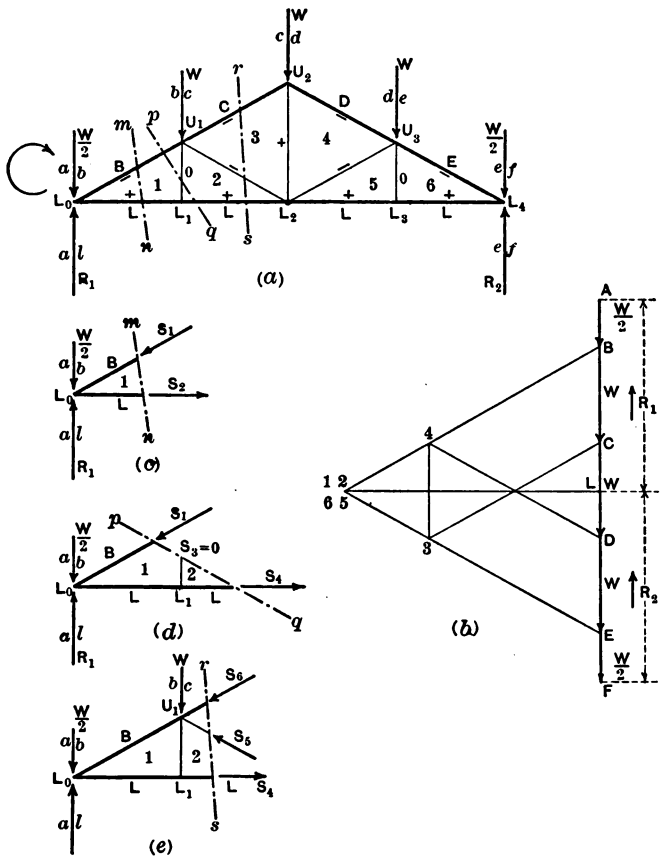


FIG. 47.

Example. Roof truss of Fig. 47 (same as Fig. 43). Same loads as before. As the reactions will be the same as given in Fig. 43 (b), this construction will not be repeated. Lay off the load-line $A\ F$ with reactions $F\ L$ and $L\ A$ at L_4, and L_0 respectively. Now pass a section

cutting two members at one end of the truss. This section is *m–n* and the members cut are $L_0 U_1$ and $L_0 L_1$. Fig. 47 (*c*) shows the portion of the structure to the left of the section free. The unknowns are S_1 and S_2, the stresses in $L_0 U_1$ and $L_0 L_1$. A simple force polygon, $L A B I L$, of Fig. (*b*) gives the unknown stresses. The kind of stress is found in the same way as in the method of successive joints. In passing around a joint the forces should be taken in the order shown by the circular arrow to the left of Fig. (*a*).

Now pass the section *p–q* cutting $L_0 U_1$, $U_1 L_1$ and $L_0 L_1$. The portion of the structure to the left of the section, with all forces applied is shown in Fig. (*d*). S_3 and S_4 are the unknown stresses. The value of S_1 has already been determined, and, together with $\frac{W}{2}$ and R_1, constitute the known forces. Since the forces are in equilibrium the force polygon must close. The portion $L A B I L$ (Fig. (*b*)) of the polygon is already drawn; 1–2 and 2 L drawn parallel to their respective members closes the polygon and determines the stresses in the pieces. The stress in $U_1 L_1$ is zero and that in $L_1 L_2$ is equal to the stress in $L_0 L_1$. Again pass a section *r–s*. Fig. (*e*) shows the portion of the structure to be considered. The members cut are $U_1 U_2$, $U_1 L_2$ and $L_1 L_2$. The stresses in $U_1 U_2$ and $U_1 L_2$, S_6 and S_5 respectively are unknown, while S_4, the stress in $L_1 L_2$, the reaction R_1, and the joint loads at L_0 and U_1 are known. Complete the force polygon for these forces, 2 $L A B C$ 3 2 of Fig. (*b*). In constructing this force polygon start with the farthest known force to the left, S_4, given by 2 L of Fig. (*b*). Starting at point, 2, S_4 passes from 2 to L. The reaction passes from L to A, then, in order $\frac{W}{2}$, W, S_6 and S_5 pass, respectively, from A to B, B to C, C to 3, and 3 to 2, finally closing on point 2, the point of beginning.

In the same way pass successive sections until the stresses in all members have been determined. While the above is a different method from that following in the method of successive joints of Fig. 43, yet the resulting diagram is precisely the same as obtained by that method, the force polygon for any joint being given directly in the figure. Moreover, if we pass any section whatever through the structure, the polygon of the forces acting upon either portion will be given by the diagram.

50. *Special Application.*—In case it is necessary to determine the stresses in but one or two members of the structure when under a given loading, the preceding method necessitates the finding of all the stresses up to the ones in question, and thus is a much longer process than the analytical method of sections. If there are no external forces applied to the structure between one of the abutments and the section cutting the members in question, as is often the case, the desired stresses can readily be found by graphics as follows: In Fig. 48 suppose the stresses

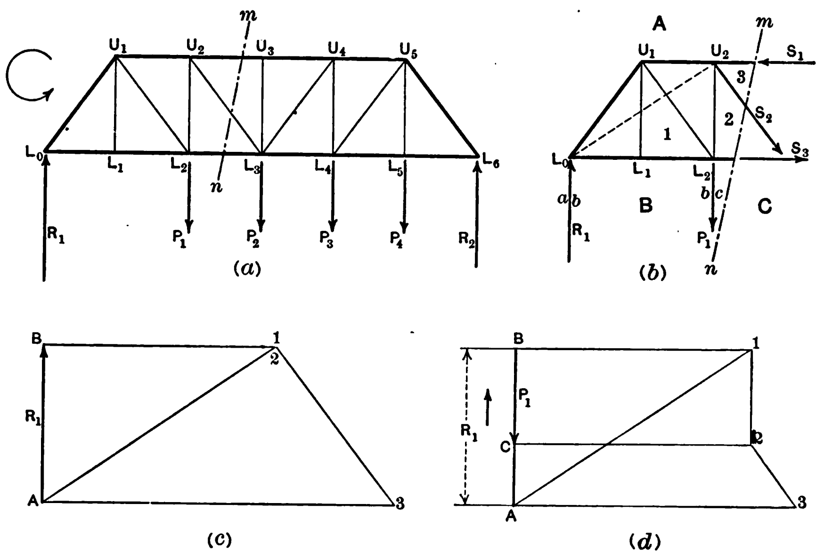


FIG. 48.

are desired in the members cut by the section *m–n* and that there are no loads to the left of this section (P_1 is not at present considered). Determine R_1 by any convenient method. Fig. (*b*) shows the portion to the left of the section with stresses S_1, S_2, and S_3 indicated. Now these stresses, with R_1, form a balanced system whose equilibrium is independent of the shape of the structure acted upon. We may therefore substitute the triangle $L_0\,U_2\,L_2$ for the actual structure to the left of $U_2\,L_2$ without changing the stresses S_1, S_2, and S_3. Let this be done, and proceed to construct the force diagram for the modified structure, beginning at L_0. In Fig. (*c*), *A B* 1 *A* is the diagram for L_0. *B*–1 is

the stress S_3. The stresses S_1 and S_2 are then found as in Art. 49 by closing the polygon $A\ B\ 1\ 2\ 3\ A$. $2–3 = S_2$ and $3–A = S_1$.

If there be a load P_1 at the joint L_2, adjacent to the section, the triangle $L_0\ U_2\ L_2$ may still be used. Fig. (*d*) gives the construction. $A\ B\ 1$ gives the stresses at L_0 and $1\ B\ C\ 2\ 1$ gives the stresses at L_2. $C–2 = S_3$. Then S_1 and S_2 are obtained by closing the polygon for $R_1\ P_1\ S_3$, S_2 and S_1. It is $A\ B\ C\ 2\ 3\ A$. $2-3 = S_2$ and $3-A = S_1$. Notice that $A\ 1\ 2\ 3\ A$ is also the diagram for joint U_2.

CHAPTER III

ANALYSIS OF ROOF TRUSSES

SECTION I.—FORMS OF TRUSSES

51. A few of the standard forms of trusses are shown in Fig. 1.

The Howe truss of Fig. (*a*) is usually made of wood except the vertical tension members, which are steel rods. The diagonal web members are all in compression, which makes this form of truss very economical for construction in wood. The Pratt truss, Fig. (*b*), is usually made of steel. In this truss the verticals take compression and the diagonals take tension. The Fink truss of Fig. (*c*) is a very economical form of roof truss. It is usually constructed of steel with riveted joints. In this form the compression members are the shortest members in the truss. It is used more than any other form of truss for moderate spans. The Fan truss, Fig. (*d*), is a form of Fink truss in which the top chord has been divided into shorter lengths in order to provide supports for purlins which would not come at panel points in the truss of Fig. (*c*). The French truss of Fig. (*e*) is also another form of Fink truss. In this truss the bottom chord is cambered for the sake of appearance or headroom in case of steep pitches. The effect of the camber in the bottom chord is to increase the stresses in the members, but at the same time the lengths of the web members are decreased. The result is that this form of truss requires about the same amount of material as the truss of Fig. (*c*). The trusses of Figs. (*f*) and (*g*) are used for flat roofs. In short spans the joints are riveted, and in longer spans pin-connected joints are used. The truss of Fig. (*g*) is used where counterbracing is to be placed in the panels. The saw-tooth construction of Fig. (*h*) is used for mill buildings. The shorter leg usually has a glass skylight, facing toward the north to avoid the direct light of the sun.

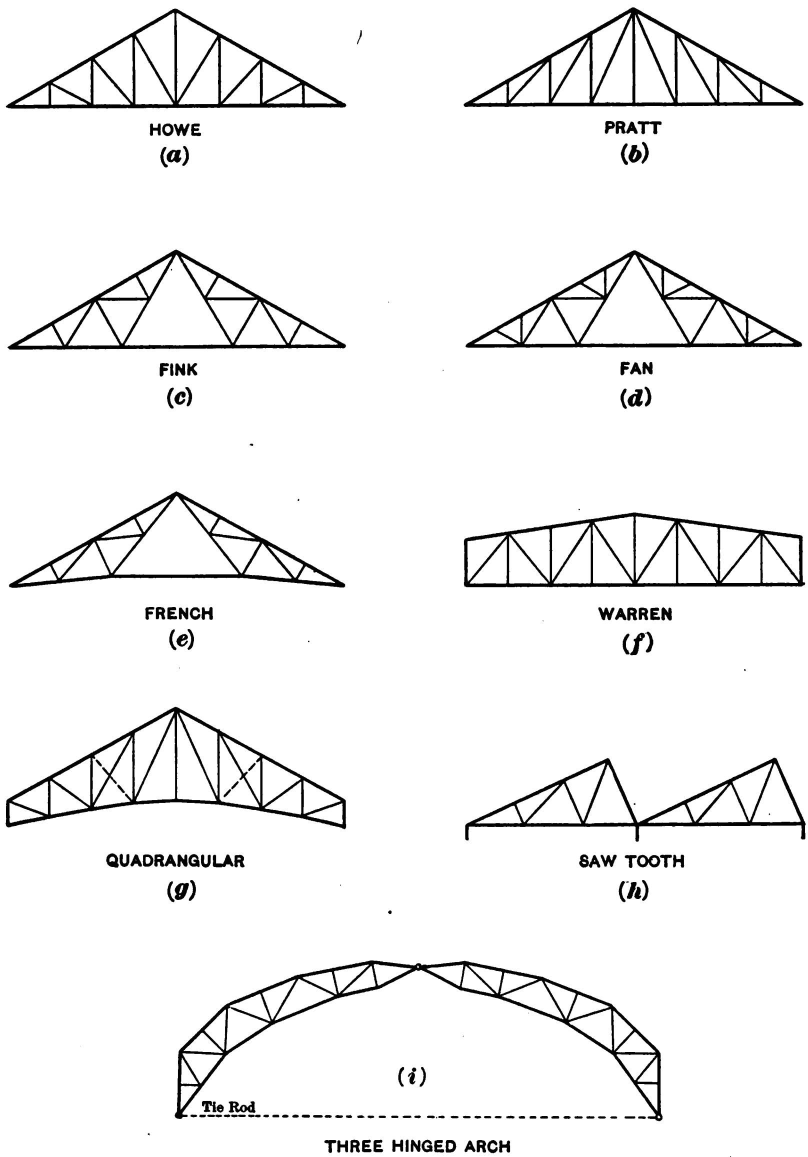

HOWE
(a)
PRATT
(b)
FINK
(c)
FAN
(d)
FRENCH
(e)
WARREN
(f)
QUADRANGULAR
(g)
SAW TOOTH
(h)
(i)
Tie Rod
THREE HINGED ARCH

FIG. 1.

The arch truss of Fig. (*i*) is used for long spans, such as railway train sheds, auditoriums, and drill halls. In the form shown, called the three-hinged arch, hinges are placed at the two abutments and also at the centre. The reactions can be determined by the methods of Arts. 44 and 45. If an arch has no hinges, or has but two hinges at the abutments, the stresses depend upon distortion as well as upon the static load. These forms are treated in Part II. In some cases where heavy abutments would be required to resist the horizontal thrust of the reactions because of poor foundations, the thrust is taken up by tie-rods under the floor, the foundations taking only the vertical components of the reactions.

Section II.—Loads and Reactions

52. Dead Load.—The dead or fixed load supported by a truss is made up of: the weight of the truss itself; the roof, including roof-covering, sheeting, rafters, and purlins; and sometimes the weight of ceilings, floors, and other loads suspended from the trusses. The roof being designed first, its weight can be directly computed, as can also be the weight of ceilings and floors. The weights of materials used for roofs, ceilings, floors, etc., can be found in tables given in Architects' and Builders' Pocketbook, by F. E. Kidder.

From these tables the weights in pounds per square foot of roof are given as follows:

Shingling—tin, 1 pound; wooden shingles, 2 to 3 pounds; slate, 8 to 10 pounds; tile, 8 to 20 pounds.

Roof coverings—corrugated iron, 1 to 3 pounds; gravel and felt, 5 to 6 pounds.

Rafters—1.5 to 5 pounds.

Purlins—wooden, 1.5 to 3 pounds; steel, 1.5 to 4 pounds.

Sheeting—1 inch thick, 3 to 5 pounds.

Plastered ceilings—10 pounds per sq. ft. of ceiling.

Total for roof covering—5 to 30 pounds.

The weight of the truss can only be approximated. Various formulas have been devised, based on actual weights of existing trusses. The conditions vary so much that any formula can give only an approximation of the true weight. When the structure has been designed,

its weight can be calculated, and if the calculated and assumed weights differ by an unreasonable amount, the structure must be recalculated, using the weight as found from the design.

From an investigation of trusses designed for the purpose, Professor N. Clifford Ricker has deduced the following formula: *

$$w = \frac{l}{25} + \frac{l^2}{6,000}. \qquad (1)$$

Where w = weight of truss in pounds per square foot of horizontal projection, and l = span in feet. The type of truss used was that of Fig. 1 (a), material long-leaf yellow pine. The span lengths used varied from 20 to 200 feet, the height of truss was taken in all cases as one-quarter of the span length.

While the above equation was derived with reference to a wooden truss of the form Fig. 1 (a), it has been found to give approximate values which can be used for the preliminary design of trusses of the type of Figs. 1 (b), (c), (d), and (e), when constructed either of wood or steel.

Other formulas which have been proposed are given below:

$w = \frac{3}{4}\left(1 + \frac{l}{10}\right)$ M. A. Howe. For wooden and steel trusses.

$w = \frac{1}{2}(1 + 0.15\,l)$ H. S. Jacoby. For wooden trusses.

$w = 0.06\,L + 0.6$ for heavy loads; $w = 0.04\,L + 0.4$ for light loads. Chas. E. Fowler. For Fink trusses.

$w = \frac{P}{45}\left(1 + \frac{l}{5\sqrt{A}}\right)$ M. S. Ketchum. For steel mill building trusses.

In the above formulas, w = weight of truss in pounds per square foot of horizontal projection, l = span in feet, A = distance between centres of trusses in feet, and P = capacity of truss in pounds per square foot of horizontal projection. Very complete tables of weights of trusses, for different span lengths are given in Part III of Building Construction, by F. E. Kidder, from which Tables I and II have been taken.

* See Bulletin No. 16, Illinois Engineering Experiment Station, August, 1907.

TABLE I.

WEIGHT OF WOODEN ROOF TRUSSES IN POUNDS PER SQUARE FOOT OF ROOF SURFACE.

Span.	Pitch of Roof.			
	½	⅓	¼	Flat.
	lbs.	lbs.	lbs.	lbs.
Up to 36 feet	3	3½	3¾	4
36 to 50 feet............	3¼	3¾	4	4½
50 to 60 feet............	3½	4	4½	4¾
60 to 70 feet............	3¾	4½	4¾	5¼
70 to 80 feet............	4¼	5	5½	6
80 to 90 feet............	5	6	6½	7
90 to 100 feet...........	5¾	6¾	7	8
100 to 110 feet...........	6½	7½	8	9
110 to 120 feet...........	7	8½	9	10

TABLE II.

WEIGHT OF STEEL ROOF TRUSSES IN POUNDS PER SQUARE FOOT OF ROOF SURFACE.

Span.	Pitch of Roof.			
	½	⅓	¼	Flat.
	lbs.	lbs.	lbs.	lbs.
Up to 40 feet.........	5.25	6.3	6.8	7.6
" " 50 feet..........	5.75	6.6	7.2	8.0
" " 60 feet..........	6.75	8.0	8.6	9.6
" " 70 feet..........	7.25	8.5	9.2	10.2
" " 80 fect..........	7.75	9.0	9.7	10.8
" " 100 feet..........	8.5	10.0	10.8	12.0
" " 120 feet....	9.5	11.0	12.0	13.2
" " 140 feet.	10.0	11.6	12.6	14.0

The weight of trusses of the form of Figs. 1 (*f*), (*g*), and (*h*), for short spans can be estimated by the above Tables. For long spans each case will require special consideration. For the arch truss of Fig. 1 (*i*), the weight of truss to be used in the preliminary design can be found best by comparison with existing structures of the same size.

The effect of machinery or shafting loads on the weight of trusses depends so much upon the special case under consideration that a formula cannot be expected to cover a general case. The amount to be added is to be determined by the experience of the designer.

53. **Live Load.**—The live or variable load consists of, (*a*), the wind load, (*b*), snow load, (*c*), floor loads, if any.

(*a*) *The Wind Load.*—The pressure of the wind upon a plane surface normal to its direction is found to be closely proportional to the square of the wind velocity. Experiments made by Professor C. F. Marvin on Mt. Washington * gave results agreeing closely with the formula $P = 0.004\ V^2$, where P = pressure in pounds per sq. ft., and V = velocity of wind in miles per hour. Recent experiments made at the Eiffel Tower and at the National Physical Laboratory of England gave results agreeing very closely with each other but differing somewhat from those of Professor Marvin. These results accord very closely with the formula

$$P = 0.0032\ V^2,$$

for square surfaces from 10 to 100 sq. ft. in area. For smaller areas the pressures were somewhat less. Adopting the latter formula gives pressures for various wind velocities as follows:

Wind Velocities, Miles per Hour.	Pressure, Pounds per Sq. Ft.
20	1.2
40	5.1
60	11.5
80	20.5
100	32.0

The wind velocities in severe storms in the United States rarely exceed 60 miles per hour, excepting in the case of hurricanes or tornadoes. In tropical hurricanes reaching into the southern border of the United States, wind velocities of from 80 to 140 miles per hour have been recorded. In estimating the pressure for large areas, or for small areas constituting parts of a large structure, such as a long-span bridge, the variable character of the wind should be taken into account. While any small part of a structure is likely to be exposed to the maximum

* Engineering News, Dec. 13, 1890.

velocity and pressure no very large area will be exposed to such pressures at the same instant. The maximum average for large areas will thus be considerably less than for small areas, such as were used in the experiment referred to. Some experiments made by Sir Benjamin Baker during the erection of the Forth Bridge* showed that the ratio of the unit pressure upon an area of 1½ sq. ft. to that upon an area of 300 sq. ft., varied from 1.3 to 2.5, averaging about 1.5. The highest pressure recorded during the seven years over which the observations extended was 41 lbs. per sq. ft. upon the smaller surface and 27 lbs. upon the larger. The gales experienced in that vicinity are very severe.

The wind velocities in tornadoes have never been measured, but the pressures exerted have been approximately determined in a few instances by a study of the effect on certain structures. In the St. Louis tornado of 1896 there was evidence of a pressure of 60 lbs. per sq. ft. on a length of 180 ft.† A study of the effect of tornadoes, by C. Shaler Smith and others, leads to the conclusion that a pressure exceeding 30 lbs. per sq. ft. is very improbable over a space as wide as 150 or 200 feet, and that generally a greater pressure than this does not extend over a path wider than 60 feet.‡

Considering the above data and the known stability of structures properly designed under modern specifications, a maximum pressure of 30 lbs. per sq. ft. would appear ample for areas of any considerable size; and in localities not subject to tornadoes or hurricanes, or in protected locations, a maximum pressure of 20 to 25 lbs. per sq. ft. is sufficient. The usual assumption for roofs and exposed buildings is 30 lbs.

The effect of wind is to cause not only a pressure on the windward side of a structure, but also a suction effect on the leeward side of a roof or building, resulting in an outward and upward pressure. In tornadoes the effect is very great and is measured by a marked drop in the barometer in a short distance. The resulting bursting effect is obvious in many wrecked structures. This action should be provided against, at least to an extent equal to that for direct pressure, by proper anchorage of trusses and bracing of frames against outward as well as inward pressures.

* Engineering, Feb. 28, 1890.

† Trans. Am. Soc. C. E., Vol. XXXVII, p. 221.

‡ Trans. Am. Soc. C. E., Vol. LIV, 1905, p. 37.

The pressures above considered are those exerted by the wind acting normally to the exposed surface. In estimating the pressure on roofs and other inclined surfaces the direction of the wind is assumed to be horizontal and the normal component of the pressure calculated. Its tangential component is small and is neglected. The normal component is calculated from empirical formulas derived from experiments. Two formulas are in use.

Hutton's formula is:

$$P_n = P \sin \alpha^{\,1.84 \cos \alpha - 1}. \qquad (2)$$

Duchemin's formula is:

$$P_n = P \frac{2 \sin \alpha}{1 + \sin^2 \alpha}. \qquad (3)$$

In these formulas

α = angle of inclination with the horizontal,
P_n = normal component of pressure,
P = pressure per sq. ft. on a vertical surface.

Duchemin's formula gives somewhat the higher values and is considered to be the more reliable. The normal component for various angles for pressures of 20 and 30 lbs., calculated by this formula, are as follows:

TABLE III.

WIND LOAD IN POUNDS PER SQUARE FOOT OF ROOF SURFACE.

Inclination. α	Normal Pressure, P_n, Pounds per Square Foot.	
	P = 30 lbs.	P = 20 lbs.
5°	5.1	3.4
10°	10.1	6.7
15°	14.6	9.7
21° 48′ (1/5 pitch)	19.8	13.1
26° 34′ (1/4 pitch)	22.4	14.0
30°	24.0	16.0
33° 41′ (1/3 pitch)	25.5	17.0
40°	26.7	17.8
45° (1/2 pitch)	28.3	18.9
60° and above	30	20

Pressures on other than plane areas are variously estimated about as follows: On cylindrical surfaces, 60 to 66-2/3 per cent. of that on plane areas; on octagonal prisms, 70 per cent.; on concave-shaped areas 125 to 150 per cent.

(*b*) *The Snow Load.*—The snow load is a variable quantity, depending upon the latitude and the humidity. Dry, freshly fallen snow weighs about 8 pounds per cubic foot, and may attain a depth of 3 feet on flat roofs. Packed snow will weigh about 12 pounds per cu. ft., while saturated snow, or snow mixed with hail or sleet, may weigh as much as 30 pounds per cu. ft. Such snow will seldom be found more than about 16 inches deep. It is generally assumed that a roof will not be subjected to the maximum wind and snow loads at the same time, but as a sleet or heavy wet snow storm may often be followed by a high wind, some allowance should be made for a part, at least, of the snow or sleet remaining on the roof. This allowance is usually one-half the maximum snow load. The following table, taken from Part III of Building Construction, by F. E. Kidder, gives a suitable snow allowance for various conditions.

TABLE IV.

ALLOWANCE FOR SNOW IN POUNDS PER SQUARE FOOT OF ROOF SURFACE.

Location.	Pitch of Roof.				
	1/2	1/3	1/4	1/5	1/6 or less
	* †	* †	* †		
Southern States and Pacific Slope	0– 0	0– 5	0– 5	5	5
Central States	0– 5	7–10	15–20	22	30
Rocky Mountain States	0–10	10–15	20–25	27	35
New England States	0–10	10–15	20–25	35	40
Northwest States	0–12	12–18	25–30	37	45

* For slate, tile, or metal roofs. † For shingle roofs.

Snow load need not be considered on roofs where the inclination to the horizontal is greater than 45° to 60°, depending upon the smoothness.

(*c*) *Floor Loads and Other Loads.*—Live loads for floors are hard to classify. Each case must be considered in detail in determining the loads to be used. The building laws of large cities specify certain loads,

by which the engineer must be governed. The table below gives a few of these values:

TABLE V.

FLOOR LOADS SPECIFIED IN BUILDING LAWS OF VARIOUS CITIES.

Live Loads for Floors in Different Classes of Buildings, Exclusive of the Weight of the Materials of Construction.	New York. 1906.	Chicago. 1906.	Philadelphia. 1906.	Boston. 1906.
	Pounds per Square Foot.			
Dwellings, Apartment Houses, Hotels, Tenement Houses, or Lodging Houses	60	50	70	50
Office Buildings—First Floor	150	100	100	100
Office Buildings—above First Floor	75	100	100	100
Schools or Places of Instruction	75	75	...	80
Buildings for Public Assembly	90	100	120	150
Buildings for Ordinary Stores, Light Manufacturing, and Light Storage	120	100	120	...
Stores for Heavy Materials, Warehouses, and Factories	150	...	150	250

Where cranes, shafting, machinery, or other concentrated loads are carried directly by the trusses, or on floors supported by trusses, their exact weights can be obtained and applied at the proper places.

(*d*) *Combined Effect of Snow and Wind Loads.*—In some cases an allowance is made for snow and wind by assuming their combined effect to be equivalent to that of a uniform vertical load applied over

TABLE VI.

ALLOWANCE FOR WIND AND SNOW COMBINED, IN POUNDS PER SQUARE FOOT OF ROOF SURFACE.

Location.	Pitch of Roof.					
	60°	45°	1/3	1/4	1/5	1/6
Northwest States	30	30	25	30	37	45
New England States	30	30	25	25	35	40
Rocky Mountain States	30	30	25	25	27	35
Central States	30	30	25	25	22	30
Southern and Pacific States	30	30	25	25	22	20

the whole truss. It has been found that for ordinary wooden and steel trusses, whose inclination to the horizontal is not more than 45°, this method gives results which are sufficiently accurate. A table giving the amounts of such loads for various conditions is given in Part III of Building Construction, by F. E. Kidder.

These values are given in the preceding table.

54. Apex Loads.—The loads carried by roof trusses are assumed as applied at the joints of the upper chord. The weight of the roof, and the wind and snow loads are transferred to the truss by means of the purlins. In large roofs the purlins should, if possible, be placed upon the trusses at the joints; but if it is necessary to place them between joints, the members of the upper chord supporting them must be designed to resist as a beam as well as a compression member of the truss. The weight of the truss is usually assumed as divided between the panel points of the top chord. For large trusses part of the load should be distributed over the bottom chord also. Ceiling or floor loads carried by a truss are distributed among the panel points of the loaded chord. Machinery or shafting loads are assumed or applied at some part of the truss. If such loads are suspended from a truss member, this member must be designed as a beam as well as a truss member.

The snow and roof loads being vertical and uniformly distributed over each panel, the joint loads are each equal to one-half the sum of the adjacent panel loads. Thus the load at *b*, Fig, 2, is equal to one-half the panel load on *b c* plus one-half the panel load on *a b*. The snow load on the panel *a b* is, of course, less *per square foot of roof* than on *b c*. The wind load at *b* is equal to one-half the wind load on *b c* combined with one-half the wind load on *a b*, the load on each panel being normal to the surface. If all panels in one-half the truss lie in the same plane and are equal, then all joint loads are equal.

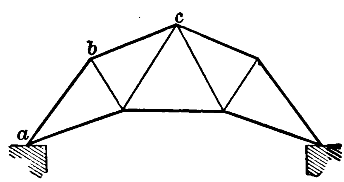


FIG. 2.

The above applies only when the purlins are placed at joints. If placed at intermediate points, the loads on the purlins are found as above and divided between adjacent joints in the inverse ratio of the distance of these joints from the purlins.

In roofs of ordinary span it is usual to attach the roof covering directly to small angle-iron purlins. In this case the roof load may be treated as a uniform load upon the truss, both in getting apex loads and in computing the bending moment in the upper chord.

55. **Reactions.**—For snow and dead loads both reactions are vertical. For wind load the reactions depend upon the manner of supporting the truss. If both ends are fixed the wind reactions are indeterminate, as four unknowns are in question—the amounts and directions of two forces. To determine the unknowns a fourth independent equation, in addition to the three equilibrium equations, must be used. This fourth equation is obtained by making certain assumptions as to the manner in which the horizontal components of the forces are divided between the two abutments. One of two assumptions is usually made, either (*a*), that the horizontal components of the two reactions are equal, or (*b*), that the direction of the reactions are parallel to the resultant wind load. Of these two, the former is probably the more correct assumption, and the one which will be used hereafter in the solution of problems under this case. If one end is free to move, i.e., on rollers or supported on a rocker, the reaction at this end is vertical and that at the fixed end follows from the analysis. If one end be fixed and the other merely supported upon a smooth iron plate, the reaction of the free end may have a horizontal component equal to the vertical component multiplied by the coefficient of friction, which is about 1/3.

SECTION III.—ANALYSIS

56. **General Methods.**—The analysis of a roof truss may be divided into two parts: (*a*) the determination of the external forces—loads, and reactions; (*b*) the determination of the internal forces—the stresses in the members.

In determining the external forces, the loads at the joints are first calculated from the data in Art. 52. Then, by the methods of Arts. 44 and 45, the remaining external forces—the reactions—are found. It will usually be found convenient to determine reactions by algebraic methods, because of the ease of calculation and also because superior accuracy can be obtained by this method. In cases where the calcula-

tion of lever arms is complicated, a large scale drawing of the truss can be made and the desired distances scaled from layout.

For the determination of the internal forces, or stresses, the graphical method of stress calculation will usually be the more convenient. As there are but three or four different possible loadings for roof trusses, it will be necessary to draw but one diagram for each loading. The loadings for which the stresses are usually determined are the dead load, the snow load, and the wind load, which is considered as applied first to one-half, and then to the other half of the truss, due to the fact that the wind may blow from either direction. As the dead and snow loads are vertical loads, considered as uniformly distributed, the panel loads will be proportional to the loadings. Then, if the stresses due to dead load are found by a stress diagram, those due to the snow load can be determined by direct ratio from the dead-load stresses without drawing a separate diagram. As the wind loading is usually an unsymmetrical loading, and not always the same for the two sides of the truss, separate wind diagrams must be drawn for both directions of the wind. Then, by a combination of the stresses thus determined, the maximum possible tension or compression to be resisted by any given member can be determined. In the following articles the complete analysis of a few standard forms of trusses will be given.

57. Analysis of a Howe Roof Truss.—A complete analysis of a truss of the form of Fig. 1 (*a*) will now be made. Fig. 3 (*a*) shows the truss to be analyzed. The following dimensions and conditions will be taken: Span, 50 ft.; rise, 12.5 ft.; distance between trusses, 16 ft.; roof divided into six equal panels; ends fixed. Length of horizontal projection of a top-chord panel will be $50 \div 6 = 8.33$ ft. The angle will be $26° 34'$. The length of the top-chord panel will then be $8.33 \times \sec\theta = 9.3$ ft.

From eq. 1, Art. 52, the weight of the truss in pounds per sq. ft. of horizontal area may be taken as $\frac{l}{25} + \frac{l^2}{6{,}000} = \frac{50}{25} + \frac{2{,}500}{6{,}000} = 2.4$ pounds. As the horizontal projection per panel is 8.33 ft. and the distance between trusses is 16 ft., the panel load due to the weight of the truss will be $8.33 \times 16 \times 2.4 = 320$ pounds. The roof covering will be assumed as shingles and sheathing carried by rafters and purlins. From the data given in Art. 52, the weight of the roof covering will be taken as 10 pounds per sq. foot of roof, made up of the

following items: Shingles, 2.5 pounds; sheathing, 3.0 pounds, rafters, 1.5 pounds; purlins, 3.0 pounds. As the top-chord panels are 9.3 feet long, the panel load due to the roof covering will be 9.3 ×

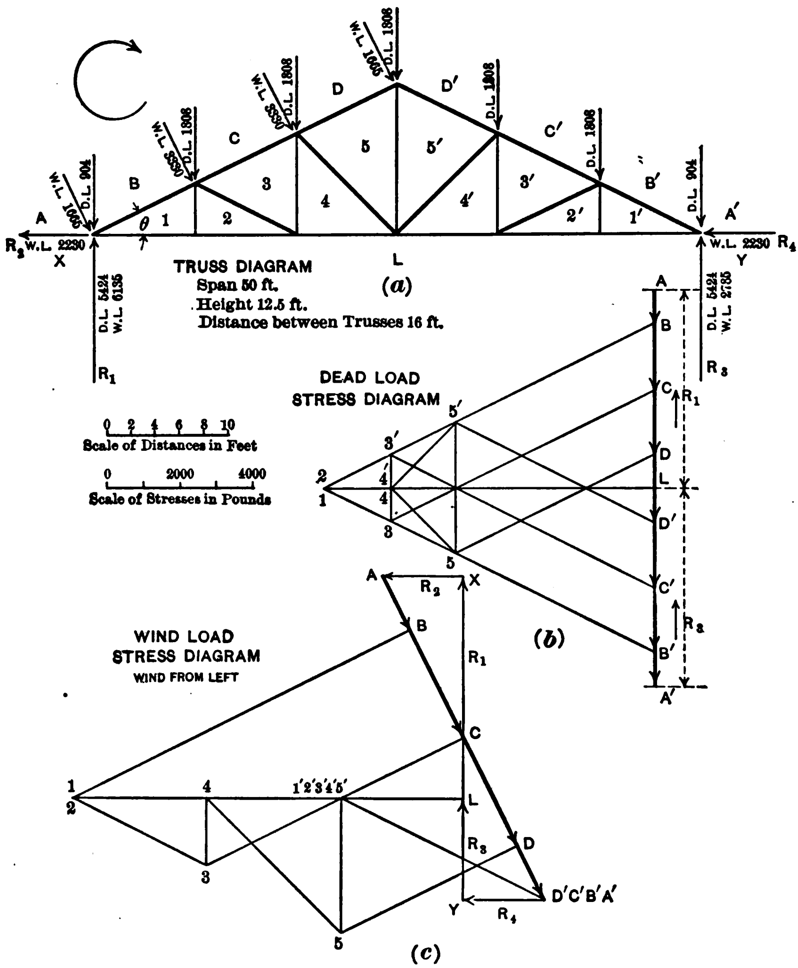


FIG. 3.

16 × 10 = 1,488 pounds. The total panel load due to dead load will be 320 pounds for the weight of the truss and 1,488 pounds for the covering, a total of 1,808 pounds.

The maximum weight of snow to be provided for is given in Table IV of Art. 53. The truss in question has a pitch (ratio of height to span length) of one-quarter. For a shingle roof the snow load for the Central States is given as 20 pounds per square foot of roof area. The snow-panel load is then $9.3 \times 16 \times 20 = 2{,}976$ pounds. The minimum weight of snow will be taken as one-half the maximum, giving a panel load of 1,488 pounds.

The wind load as given in Table III of Art. 53, for a truss of ¼ pitch, is 22.4 pounds per sq. ft. of roof, assuming the wind pressure as 30 pounds per sq. ft. of vertical projection. The panel load due to wind will then be $22.4 \times 9.3 \times 16 = 3{,}330$ pounds.

After drawing the truss carefully to scale, as shown in Fig. 3 (*a*) (the scale used for this case was 5 feet to an inch), we proceed to draw the diagram for dead load, (Fig. 3 (*b*)). Each joint load is 1,808 pounds, except the loads at the end joints which are half loads, or 904 pounds. These loads are laid off to form the load-line $A\,A'$. Since the loads are symmetrical about the centre of the truss, the reactions are equal, and are represented by $A'\,L$ and $L\,A$, each equal to $\frac{1}{2}\,A\,A'$. Beginning at the left end of the truss, the diagram is drawn exactly as in Fig. 43, Art. 47. The scale used in this case was 2,000 pounds to one inch. The amount and kind of stress in each truss member is given in Column 1 of the table of stresses; they were readily scaled off to the nearest ten pounds. The diagram for maximum snow load will be a figure similar to the one for dead load, and the stresses in the two cases will be proportional to the corresponding panel loads. Therefore, if we multiply each dead-load stress by $\frac{2{,}976}{1{,}808}$, we will have the corresponding snow-load stress. These are best found with the slide rule. Column 2 of the table gives the maximum snow-load stresses. The minimum snow-load stresses are one-half the maximum snow-load stresses. They are given in Column 3.

For wind stresses we must consider the wind blowing first from one side; then from the other. Since the ends of the truss are assumed as fixed, we will consider the horizontal components of the wind as equally divided between the two abutments.

Fig. 3 (*c*) is the diagram for wind from the left. The load-line is $A\,D'$. The reactions are best found algebraically. A moment equa-

tion about the left end of the truss gives $R_3 = 2{,}785$ pounds. In the same way, moments about the right abutment gives $R_1 = 6{,}135$ pounds. R_2 and R_4, the horizontal components of the reactions, are found by a resolution equation for horizontal forces. Thus $R_2 = R_4 = \frac{1}{2}(1{,}665 + 3{,}330 + 3{,}330 + 1{,}665) \sin 26° \, 34' = 2{,}230$ pounds. The reactions are then plotted in the line $D' \, Y \, L \, X \, A$, closing the polygon for external forces. Beginning at the left end of the truss, the stress diagram is readily constructed. It will be found that there are no stresses in the web members of the right half of the truss. Column 4 of the table gives the stresses as found from the diagram.

The stresses for the wind from the right can be obtained from Fig. (*c*), as the diagrams for the two directions will be similar in form but of *opposite hand.* That is, the stress in any member in one half of the truss for the wind from the right will be equal to that for the corresponding member in the other half of the truss for the wind from the left. Thus the stress in member $C' \, 3'$ for wind from the right is equal to that given by the diagram for wind from the left for member $C \, 3$. Column 5 of the table gives the stresses for wind from the right.

The stresses as obtained for the different conditions of loading, and recorded in Columns 1 to 5 of the table, must now be combined to give the maximum stresses in the members. The possible combinations are: Dead load and maximum snow load; and dead load, minimum snow load, and wind from the right or left. These combinations are made up for each member and the greatest value thus obtained is given in Column 7. In Column 8 are given the numbers of the columns used in making up the results. The table of stresses shows that the partial loading does not cause a reversal of stress in any member. For this reason it is sometimes considered sufficient to use a uniform vertical load over the whole roof surface, which shall be equivalent to the combined effect of the wind and snow. For a truss of $\frac{1}{4}$ pitch, Table VI of Art. 53, gives this load as 25 pounds per square foot of roof.

The panel load for the truss in question would then be $9.3 \times 16 \times 25 = 3{,}720$ pounds. Adding to this the dead panel load of 1,808 pounds gives a total panel load of 5,528 pounds. The stresses caused by this load were found by ratio from the dead-load stresses, and are given in Column 6 of the table of stresses. By comparing the maximum stresses as given for the partial loadings with those of Column 6 it will

TABLE OF STRESSES.

Member.	1 Dead Load.	2 Maximum Snow Load.	3 Minimum Snow Load.	4 Wind from Left.	5 Wind from Right.	6 Uniform Load.	7 Maximum Stress.	8 Columns Combined for Maximum.
B 1	− 10100	− 16620	− 8310	− 10290	− 6170	− 30880	− 28700	1 3 4
C 3	− 8065	− 13270	− 6635	− 7800	− 6170	− 24660	− 22500	1 3 4
D 5	− 6030	− 9920	− 4960	− 5320	− 6170	− 18440	− 17160	1 3 5
D' 5'	− 6030	− 9920	− 4960	− 6170	− 5320	− 18440	− 17160	1 3 4
C' 3'	− 8065	− 13270	− 6635	− 6170	− 7800	− 24660	− 22500	1 3 5
B' 1'	− 10100	− 16620	− 8310	− 6170	− 10290	− 30880	− 28700	1 3 5
L 1	+ 9020	+ 14850	+ 7425	+ 10680	+ 3270	+ 27580	+ 27125	1 3 4
L 2	+ 9020	+ 14850	+ 7425	+ 10680	+ 3270	+ 27580	+ 27125	1 3 4
L 4	+ 7190	+ 11830	+ 5915	+ 6970	+ 3270	+ 21980	+ 20075	1 3 4
L 4'	+ 7190	+ 11830	+ 5915	+ 3270	+ 6970	+ 21980	+ 20075	1 3 5
L 2'	+ 9020	+ 14850	+ 7425	+ 3270	+ 10680	+ 27580	+ 27125	1 3 5
L 1'	+ 9020	+ 14850	+ 7425	+ 3270	+ 10680	+ 27580	+ 27125	1 3 5
1 2	0	0	0	0	0	0	0	
3 4	+ 905	+ 1490	+ 745	+ 1855	0	+ 2770	+ 3505	1 3 4
5 5'	+ 3620	+ 5960	+ 2980	+ 3710	+ 3710	+ 11070	+ 10310	1 3 4 or 5
3' 4'	+ 905	+ 1490	+ 745	0	+ 1855	+ 2770	+ 3505	1 3 5
1' 2'	0	0	0	0	0	0	0	
2 3	− 2020	− 3320	− 1660	− 4130	0	− 6180	− 7810	1 3 4
4 5	− 2540	− 4180	− 2090	− 5220	0	− 7770	− 9850	1 3 4
4' 5'	− 2540	− 4180	− 2090	0	− 5220	− 7770	− 9850	1 3 5
2' 3'	− 2020	− 3320	− 1660	0	− 4130	− 6180	− 7810	1 3 5

— denotes compression. + denotes tension.

be seen that for the top- and bottom-chord members the uniform load gives greater stresses, while the partial loadings give the greater stresses in all but one web member.

58. Analysis of a Fink Truss.—Let the truss of Fig. 4 (*a*) have a span of 72 feet, rise of 24 feet, distance between trusses 16 feet, left end fixed, right end on rollers, roof divided into eight panels. The inclination of the roof to the horizontal will be $33° 41'$, or $1/3$ pitch. The length of a top-chord panel is 10.8 feet, and its horizontal projection is 9 feet. The weight of truss per sq. ft. horizontal projection is given by eq. (1), Art. 52, as 3.74 pounds.

The panel load will be $9 \times 16 \times 3.74 = 538$ pounds for weight of truss. The weight of roof covering will be assumed as corrugated steel on purlins, which will weigh 7.0 pounds per sq. ft. of roof. The panel load for roof covering will be $10.8 \times 16 \times 7.0 = 1{,}210$ pounds. The total dead panel load will then be $538 + 1{,}210 = 1{,}748$ pounds. These loads are shown in position in Fig. (*a*), half-panel loads being placed at the abutments. Next proceed to draw the dead-load diagram, Fig. (*b*). As the truss, and also the loads are symmetrical about the centre, only one-half the diagram need be drawn. The load line is $A\ L$. The stress diagram can be drawn as before until we reach joint U_2. Here three unknowns are in question, the stresses in members D 5, 5–4 and 3–4.

The unknowns at this joint can be reduced to two for the time being by replacing members 6–5 and 5–4 by the member 6–X, dotted in Fig. (*a*). The polygon for joint U_2 then becomes 3, 2, C, D, X 3. Passing now to joint U_3, the polygon is $X\ D\ E\ 6\ X$. The polygon will give the true stress in E 6, for it can be seen that the stress in E 6 will not be affected by any change in the form of the truss below joint U_3, provided the position of the loads is not altered. Again replacing members 6–5 and 5–4, and removing 6–X, pass around joint U_3, determining the true stress in D 5 and also the stress in D 6. Now pass to joint U_2, at which two unknowns now exist, the stresses in 5–4 and 3–4. The remaining stresses are readily determined as in previous cases. The stresses as scaled from the completed diagram are given in Column 1 of the Stress Table. The stress diagram of Fig. (*b*) suggests a simple method of passing around the joint U_2 without substituting member 6–X. When the panel loads are equal and symmetrically placed,

points 1, 2, 5, and 6 of the stress diagram will lie on a straight line. After the position of point 1 has been determined, points 2, 5, and 6 can be located by erecting a perpendicular to B 1. The intersection of

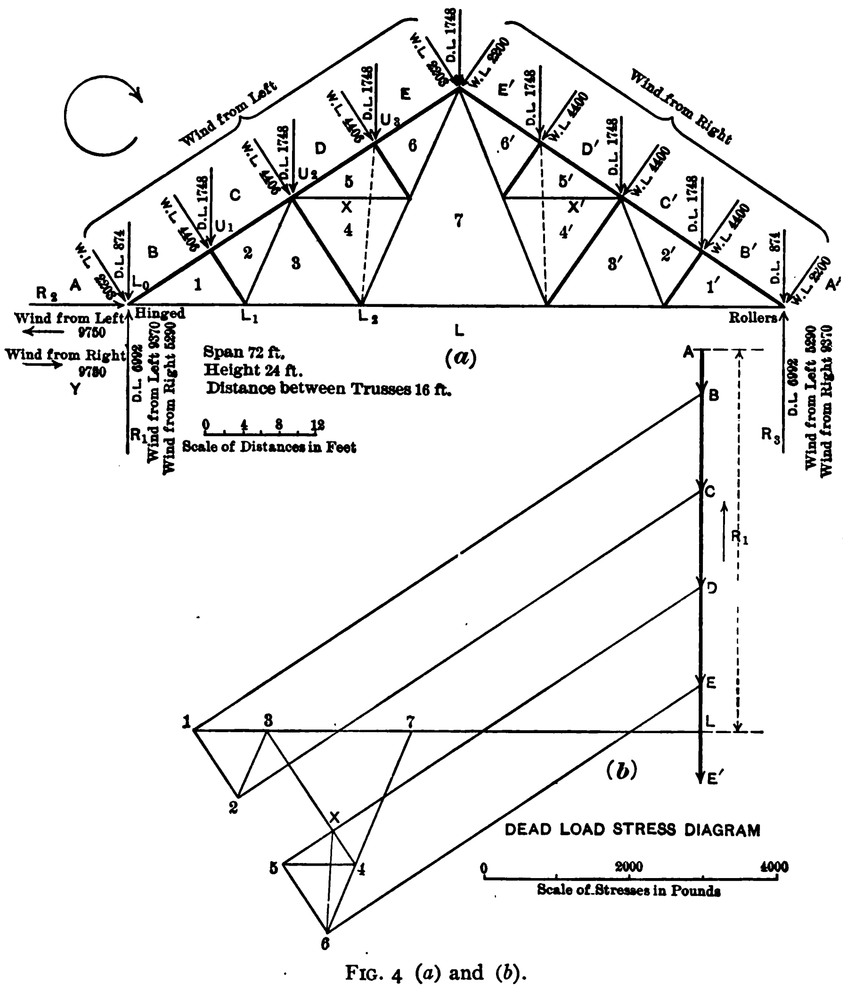


FIG. 4 (*a*) and (*b*).

this line with C 2, D 5, and E 6 will determine the stresses in the corresponding members. When these stresses are known it will be possible to pass around all the points, for only two unknowns exist at each point. This construction also holds for the wind load diagram. When the

loads are unequal this method cannot be used, as the points 1, 2, 5, and 6 will no longer be on a straight line.

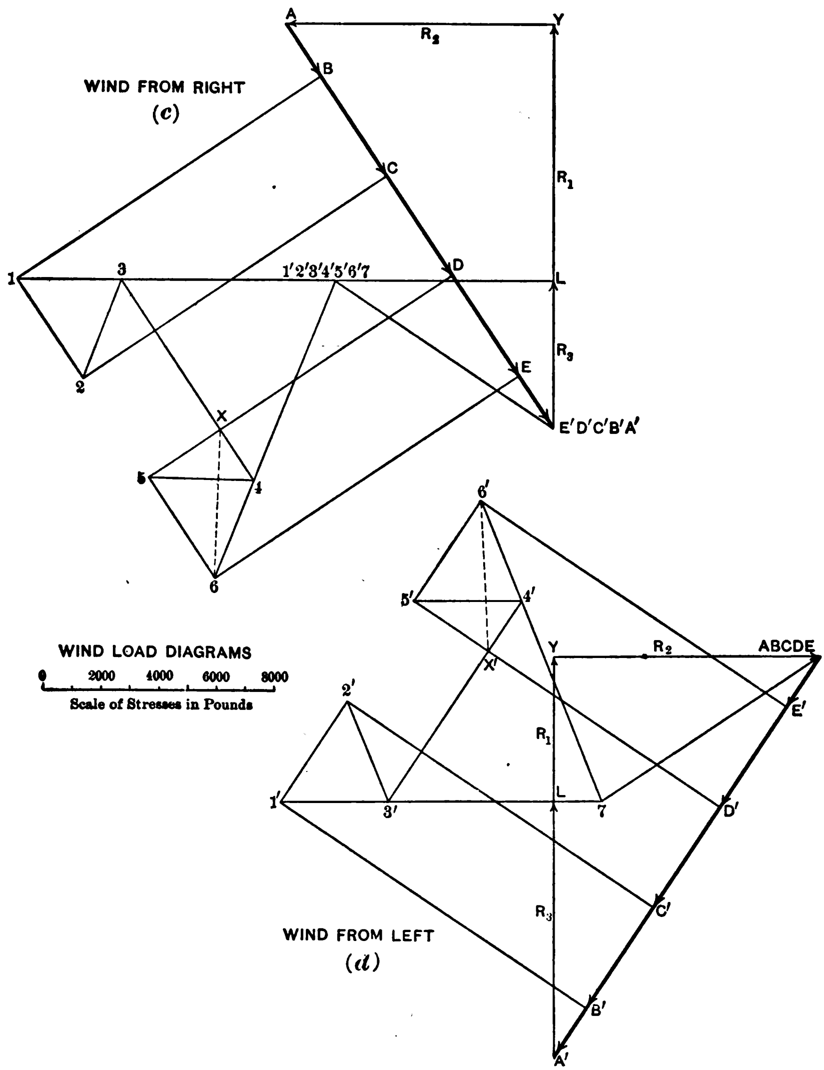


FIG. 4 (c) and (d).

The maximum and minimum snow-load stresses are determined by ratio from the dead-load stresses, and appear in Columns 2 and 3 of

the table. The maximum snow load per square foot of roof is given in Table IV of Art. 53, as 7 pounds for a truss of 1/3 pitch with a metal roof, location Central States. Panel load = 10.8 × 16 × 7 = 1,209.6 pounds. Use 1,210 pounds. The minimum panel load will be one-half the maximum, or 605 pounds.

For wind stresses we must consider the wind blowing first from one side, then from the other, since the abutment reaction at the roller end must in both cases be vertical. The stresses in the two cases will, therefore, not be symmetrical. The wind load is due to a normal pressure of 25.5 pounds per sq. ft. of roof, as given by Table III of Art. 53, assuming 30 pounds per sq. ft. on a vertical area. The total panel load is then 25.5 × 16 × 10.8 = 4,406 pounds. Fig. 4 (*c*) is the diagram for wind from the left, and Fig. 4 (*d*) is the diagram for wind from the right. The abutment reactions as calculated algebraically are given on the truss diagram of Fig. (*a*). The stresses as scaled from the diagrams are given in Columns 4 and 5 of the stress table.

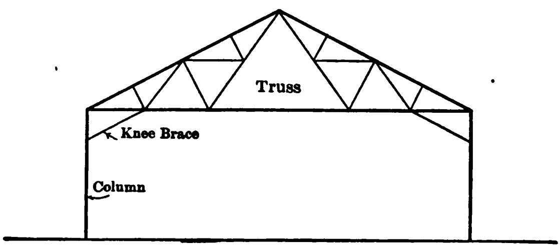


FIG. 5.

It is seen that the stresses in pieces $L\ 1$, $L\ 3$, and $L\ 7$ are compressive for wind from the right, whereas they were tensile for dead load; the ***resultant*** stresses are, however, all tensile. In Column 6 are given the maximum stresses as made up for the possible combinations of the partial loadings.

In some cases the truss considered in this article is placed on columns instead of on masonry walls. In such a case inclined members, called knee braces, are used to join the columns to the truss and form a rigid system to resist the horizontal thrust due to wind. Fig. 5 shows such an arrangement. This form of truss will be treated in Chapter VI.

TABLE OF STRESSES.

Member.	1	2	3	4	5	6	Columns Combined for Maximum.
	Dead Load.	Maximum Snow Load.	Minimum Snow Load.	Wind from Left.	Wind from Right.	Maximum Stress.	
B 1	− 11020	− 7630	− 3815	− 13420	− 9500	− 28255	1 3 4
C 2	− 10050	− 6960	− 3480	− 13420	− 9500	− 26950	1 3 4
D 5	− 9070	− 6280	− 3140	− 13420	− 9500	− 25630	1 3 4
E 6	− 8100	− 5610	− 2805	− 13420	− 9500	− 24325	1 3 4
E' 6′	− 8100	− 5610	− 2805	− 9500	− 13420	− 24325	1 3 5
D' 5′	− 9070	− 6280	− 3140	− 9500	− 13420	− 25630	1 3 5
C' 2′	− 10050	− 6960	− 3480	− 9500	− 13420	− 26950	1 3 5
B' 1′	− 11020	− 7630	− 3815	− 9500	− 13420	− 28255	1 3 5
L 1	+ 9160	+ 6340	+ 3170	+ 19700	− 1800	+ 32030	1 3 4
L 3	+ 7860	+ 5440	+ 2770	+ 15800	− 1800	+ 26430	1 3 4
L 7	+ 5250	+ 3630	+ 1815	+ 7900	− 1800	+ 14965	1 3 4
L 3′	+ 7860	+ 5440	+ 2770	+ 7900	+ 6100	+ 18530	1 3 4
L 1′	+ 9160	+ 6340	+ 3170	+ 7900	+ 10000	+ 22330	1 3 5
1 2	− 1460	− 1010	− 505	− 4400	0	− 6365	1 3 4
3 4	− 2920	− 2020	− 1010	− 8800	0	− 12730	1 3 4
5 6	− 1460	− 1010	− 505	− 4400	0	− 6365	1 3 4
5′ 6′	− 1460	− 1010	− 505	0	− 4400	− 6365	1 3 5
3′ 4′	− 2920	− 2020	− 1010	0	− 8800	− 12730	1 3 5
2′ 1′	− 1460	− 1010	− 505	0	− 4400	− 6365	1 3 5
2 3	+ 1300	+ 900	+ 450	+ 3900	0	+ 5650	1 3 4
4 5	+ 1300	+ 900	+ 450	+ 3900	0	+ 5650	1 3 4
4′ 5′	+ 1300	+ 900	+ 450	0	+ 3900	+ 5650	1 3 5
2′ 3′	+ 1300	+ 900	+ 450	0	+ 3900	+ 5650	1 3 5
4 7	+ 2600	+ 1800	+ 900	+ 7850	0	+ 11350	1 3 4
6 7	+ 3900	+ 2700	+ 1350	+ 11800	0	+ 17050	1 3 4
6′ 7	+ 3900	+ 2700	+ 1350	0	+ 11800	+ 17050	1 3 5
4′ 7	+ 2600	+ 1800	+ 900	0	+ 7850	+ 11350	1 3 5

− denotes compression. + denotes tension.

59. Analysis of a Quadrangular Truss.—Truss shown in Fig. 6 (*a*). Span of 72 feet; pitch of roof, ¼. Trusses 16 feet apart. Ends assumed as fixed. In this form of truss, the wind blowing on one side of the truss will sometimes cause a reversal of stress in the diagonal web members. To avoid this, a second member, called a counter, shown by the broken lines of Fig. (*a*), is placed in each panel where such reversal is found to occur. Thus each member will be called upon to take only one kind of stress, and only one of the members in a panel will act at a time, the stress in the other member being zero. In the truss in question, the diagonal web members will be assumed to take tension only.

The panel loads will now be determined. The top-chord-panel length is found to be 10.06 feet. The dead panel load will be due to 9.3 pounds per sq. ft. of roof for the truss, as given in Table II of Art. 52, and to 7.0 pounds per sq. ft. of roof for covering, assuming the same covering as for the Fink truss of Art. 58. The total dead panel load will be $(9.3 + 7.0) \times 10.06 \times 16 = 2{,}620$ pounds. The maximum snow load will be 15 pounds per sq. ft. of roof, as given in Table IV of Art. 53. Maximum snow panel load $= 15 \times 10.06 \times 16 = 2{,}415$ pounds. Minimum snow panel load is one-half the maximum. The wind load is given in Table III of Art. 53 as 22.4 pounds per sq. ft. of roof, assuming the wind pressure to be 30 pounds per sq. ft. of vertical area. The total wind panel load is $22.4 \times 16 \times 10.06 = 3{,}606$ pounds.

The dead-load-stress diagram, given in Fig. 6 (*b*), is constructed as before. The form of the truss was first assumed such that all the diagonal web members sloped downward and toward the centre of the truss. In passing around the successive joints, it was found, however, that if members 5–6 and 7–8 sloped as assumed, they would be in compression. As this was contrary to the condition stated above, that diagonal web members were to take tension only, these members were changed to the direction shown by the full lines of the truss diagram, Fig. (*a*), after which all were found to be in tension. The stresses as determined are given in Column 1 of the Table of Stresses. The maximum snow-load stresses were obtained by ratio from the dead-load stresses, and are given in Column 2. The minimum snow-load stresses are given in Column 3.

The stress diagram for wind from the left, Fig. 6 (*c*), was then

drawn. The ends of the truss were assumed as fixed, the horizontal components of the reactions being taken as equal. Also it was assumed that the wind on the vertical sides of the truss is carried by a self-

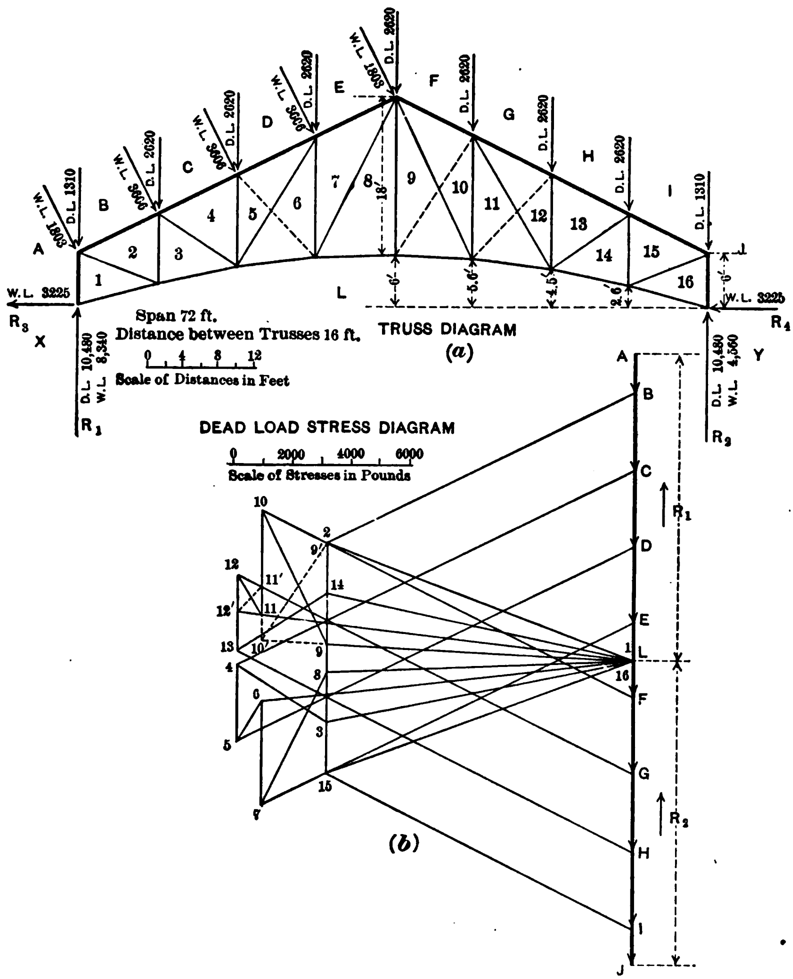


FIG. 6 (*a*) and (*b*).

supporting curtain wall. In passing around the joints of the truss it was found that the form of the truss, assuming as before that diagonal web members can take tension only, was the same for the left half of

the truss as it was for the dead load. On the right half, the dotted members, 9′–10′ (the primes indicate that the member in question is the counter), and 11′–12′ are brought into action. The stresses in these members were scaled from the diagram as 1,610 pounds tension for 11′–12′ and 905 pounds tension for 9′–10′. We must now combine these wind stresses with those due to other conditions of loading to find the total stress in the members. The possible combinations are: Dead load and maximum snow load; dead load and wind load; and dead load, wind load, and minimum snow load. As the dead-load

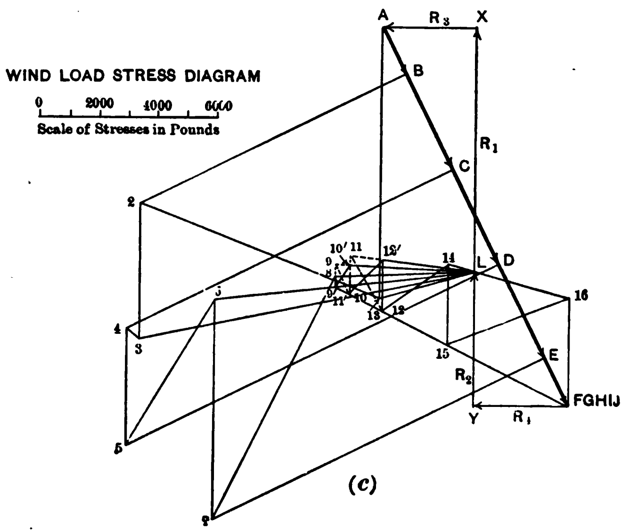


FIG. 6 (c).

diagram showed tension in the main members in these panels, it is evident that the dead-load stress in the counters will be a compression. If this compression is greater than the wind-load tension, the counter would not be called into action. Also, the minimum snow-load stresses, which will be of the same kind as the dead load, will tend to cut down the amount of tension in the counter. The maximum stress in the counter will then occur for dead load and wind load. To find the stresses in the counters due to dead load, it is necessary to revise the dead-load stress diagram, the counters 9′–10′ and 11′–12′ being supposed to act instead of the corresponding main members. The dotted

TABLE OF STRESSES.

Member.	1 Dead Load.	2 Maximum Snow Load.	3 Minimum Snow Load.	4 Wind from Left.	5 Wind from Right.	6 Dead Load and Snow Load.	7 Dead Load and Wind Load.	8 Dead Load, Minimum Snow Load and Wind Load.	9 Maximum Stress.
B 2	− 11720	− 10800	− 5400	− 10220	− 4630	− 22520	− 21940	− 27340	− 27340
C 4	− 15100	− 13920	− 6960	− 12540	− 7200	− 29020	− 27640	− 34600	− 34600
{ *D* 5	− 15100	− 13920	− 6960	− 14350	− 7200	− 29020	− 29450	− 36410	− 36410
{ *D* 6	− 14220	− 13110	− 6555		− 8410		− 22630		
E 7	− 14220	− 13110	− 6555	− 12690	− 8960	− 27330	− 26910	− 33465	− 33465
F 10	− 14220	− 13110	− 6555	− 8960	− 12690	− 27330	− 26910	− 33465	− 33465
{ *G* 12	− 15100	− 13920	− 6960	− 7200	− 14350	− 29020	− 29450	− 36410	− 36410
{ *G* 11	− 14220	− 13110	− 6555	− 8410			− 22630		
H 13	− 15100	− 13920	− 6960	− 7200	− 12540	− 29020	− 27640	− 34600	− 34600
I 15	− 11720	− 10800	− 5400	− 4630	− 10220	− 22520	− 21940	− 27340	− 27340
L 1	0	0	0	+ 3370	− 3370	0	± 3370	± 3370	± 3370
L 3	+ 10750	+ 9910	+ 4955	+ 11790	+ 955	+ 20660	+ 22540	+ 27495	+ 27495
{ *L* 6	+ 12800	+ 11800	+ 5900	+ 9000	+ 4310	+ 24600	+ 21800	+ 27700	+ 27700
{ *L* 5	+ 13600	+ 12540	+ 6270		+ 3220		+ 16820		
L 8	+ 10500	+ 9680	+ 4840	+ 4820	+ 4820	+ 20180	+ 15320	+ 20160	+ 20160
L 9	+ 10500	+ 9680	+ 4840	+ 4820	+ 4820	+ 20180	+ 15320	+ 20160	+ 20160
{ *L* 11	+ 12800	+ 11800	+ 5900	+ 4310	+ 9000	+ 24600	+ 21800	+ 27700	+ 27700
{ *L* 12	+ 13600	+ 12540	+ 6270	+ 3220			+ 16820		
L 14	+ 10750	+ 9910	+ 4955	+ 955	+ 11790	+ 20660	+ 22540	+ 27495	+ 27495
L 16	0	0	0	− 3370	+ 3370	0	± 3370	± 3370	± 3370

A 1	− 10480	− 9660	− 4830	− 9270	− 3630	− 20140	− 19750	− 24580	− 24580
2 3	− 6150	− 5670	− 2835	− 4580	− 2690	− 11820	− 10730	− 13565	− 13565
{ 4 5	− 2620	− 2415	− 1208	− 4030	o	− 5035	− 6650	− 7858	− 7858
{ 4′ 5′	− 1400	− 1290	− 645		− 1710		− 3110		
{ 6 7	− 3500	− 3230	− 1615	− 7550	− 1260	− 6730	− 11050	− 12665	− 12665
{ 6′ 7′	− 2620	− 2415	− 1208		o		− 2620		
8 9	+ 1000	+ 920	+ 460	+ 425	+ 425	+ 1920	+ 1425	+ 1885	+ 1920
{ 10 11	− 3500	− 3230	− 1615	− 1260	− 7550	− 6730	− 11050	− 12665	− 12665
{ 10′ 11′	− 2620	− 2415	− 1208	o			− 2620		
{ 12 13	− 2620	− 2415	− 1208	o	− 4030	− 5035	− 6650	− 7858	− 7858
{ 12′ 13′	− 1400	− 1290	− 645	− 1710			− 3110		
14 15	− 6150	− 5670	− 2835	− 2690	− 4580	− 11820	− 10730	− 13565	− 13565
16 J	− 10480	− 9660	− 4830	− 3630	− 9270	− 20140	− 19750	− 24580	− 24580
1 2	+ 11200	+ 10330	+ 5165	+ 8860	+ 4430	+ 21550	+ 20060	+ 25225	+ 25225
3 4	+ 3600	+ 3320	+ 1660	+ 555	+ 2740	+ 6920	+ 6340	+ 8000	+ 8000
{ 5 6	+ 1550	+ 1430	+ 715	+ 5890	− 2110	+ 2980	+ 7740	+ 8155	+ 8155
{ 5′ 6′	− 1150	− 1060	− 530	o	+ 1610		+ 460		+ 460
{ 7 8	+ 5020	+ 4630	+ 2315	+ 9320	− 1110	+ 9650	+ 14340	+ 16655	+ 16655
{ 7′ 8′	− 4050	− 3730	− 1865	o	+ 905				o
{ 9 10	+ 5020	+ 4630	+ 2315	− 1110	+ 9320	+ 9650	+ 14340	+ 16655	+ 16655
{ 9′ 10′	− 4050	− 3730	− 1865	+ 905	o				o
{ 11 12	+ 1550	+ 1430	+ 715	− 2110	+ 5890	+ 2980	+ 7440	+ 8155	+ 8155
{ 11′ 12′	− 1150	− 1060	− 530	+ 1610			+ 460		+ 460
13 14	+ 3600	+ 3320	+ 1660	+ 2740	+ 555	+ 6920	+ 6340	+ 8000	+ 8000
15 16	+ 11200	+ 10330	+ 5165	+ 4430	+ 8860	+ 21530	+ 20060	+ 25225	+ 25225

+ denotes tension. − denotes compression.

lines of Fig. (*b*) show the necessary changes. The dead-load counter-stresses are found to be 1,150 pounds compression for 11′–12′ and 4,050 pounds compression for 9′–10′. By comparing these values with the wind-load stresses as given above, it can be seen that the dead-load compression for 9′–10′ greatly exceeds the wind-load tension, so that this member will never be called into action. As a counter will not be needed in this panel, we must therefore revise the wind-load diagram in order to find the wind-load stress in the main member. The dotted line 9–10 shows this change, and the stress as scaled is 1,110 pounds compression.

Comparing the wind-load tension in 11′–12′, 1,610 pounds, with the dead-load compression, 1,150 pounds, for the same member, we see that the counter will be required in this panel, and that its stress will be 460 pounds tension. It will now be necessary to see if the counter will also be in action when the minimum snow load is acting at the same time as the wind load. The minimum snow-load stress for 11′–12′, as given by Column 3 of the Stress Table, is 530 pounds compression, which, combined with the dead-load stress of 1,150 pounds, gives a total compression of 1,680 pounds. As this stress exceeds the wind-load tension of 1,610 pounds, we see that for this condition of loading—dead load, wind load, and minimum snow load—the counter would not be in action. Therefore we see that the only condition of loading for which a counter would be required will be for dead load and wind load, and that 9′–10′ is the only counter required. It must be remembered that at the time a counter is in action, the shape of the truss is changed, and that all stresses in members of this panel must be determined for the altered shape of the truss before any combination of stress can be made. In the stress table this is done by recording each member of the second panel from the centre of the truss in two lines of the table. The first line for the member in question gives the stress when the main diagonal is in action, the second line gives the stresses for the same member when the counter is in action. Also, these stresses for the counter in action are to be used only in the combinations for dead load and wind load, for the counter acts only for this case. If it had happened that the counter acted also for other loadings, proper provision would have to be made in the table.

The combinations for the possible cases of loading were made up,

and are recorded in Columns 6, 7, and 8 of the Stress Table. Column 9 gives the maximum stress for the member as determined by a comparison of Columns 6, 7, and 8.

60. Analysis of an Arch Truss.—Let the span be taken as 150 feet, rise 46.25 feet, distance between trusses 20 feet, dimensions and arrangements of parts as shown in Fig. 7 (*a*). Assume the weight of truss and roof covering as 20 pounds per sq. ft. of roof. The panel loads are calculated by assuming the dead load to be distributed among the top-chord joints in proportion to the roof areas carried. The panel loads thus determined are given on the truss diagram. The snow load can be taken as a uniform load per sq. ft. of roof, or the load per sq. ft. for the different slopes can be taken from Table IV of Art. 53. In such a case a separate diagram for snow loads would have to be drawn as the loads would not be similar to those for dead load. The diagram for snow load has not been drawn.

The loads per sq. ft. of roof for wind will be different for the portions of the roof where the slope changes. The roof surface will be assumed to have the same slope as the top chord of the truss. Also the vertical sides will be assumed as protected by a self-supporting curtain wall which will take care of the wind at these places. In case the truss supports these side walls, the effect of horizontal wind loads would have to be considered. The wind loads to be provided for on the various portions of the roof will be found in Table III of Art. 53. For the portion *M–N*, which has a slope of 45°, the load per sq. ft. of roof will be 28.3 pounds, assuming the wind pressure at 30 pounds per sq. ft. of vertical area. On *N–R* whose slope is 21° 48′, the load will be 19.8 pounds per sq. ft. On *R–S*, slope 10° 19′, the load will be 11.0 pounds per sq. ft. For the portion *S–T*, considered as flat, the wind load will be zero. The panel loads, as calculated from the above loads per sq. ft. of roof, are given on the right half of the truss diagram of Fig. (*a*).

The dead-load-stress diagram is given in Fig. 7 (*b*). As the stresses in the two halves of the arch are symmetrical, the diagram has been drawn only on the left half. In drawing the stress diagram, the load line *A X* was first drawn. The panel load at the centre joint is assumed as divided equally between the two halves of the arch. *H X* is, therefore, equal to half the panel load at the centre. The reaction at the

left end of the arch, together with the force at the centre hinge, was then calculated by the algebraic method of Art. 44. These forces were then plotted on the load line to complete the polygon for external forces. In Fig. (*b*), $L\,A$ represents, in amount and direction, the reaction at R_1,

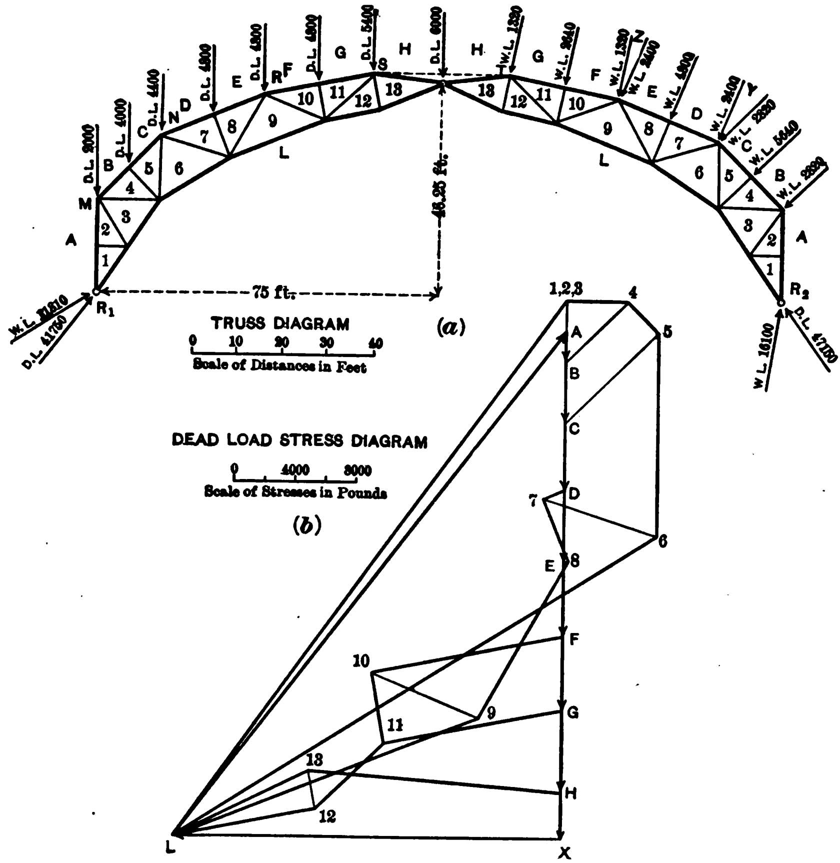


FIG. 7 (*a*) and (*b*).

and $X\,L$ represents the force at the centre hinge, which is equal to the horizontal component of the reaction at R_1.

The stress diagram was then drawn by beginning at the left abutment, taking successive joints toward the centre, as shown in Fig. (*b*). Heavy lines represent the members in compression, light lines those in tension.

In the same way, the stress diagrams for wind load were also drawn.

The reactions as calculated are given on the truss diagram. Fig. 7 (*c*) shows the stress diagram for wind from the right for the loaded half of the arch, in this case the right half. Fig. 7 (*d*) gives the stress diagram for the left, or unloaded half of the arch. In this diagram the load-line is given by the equal and opposite forces which are applied at the centre hinge and at R_1, these forces being equal to the reaction at R_1, and are shown in amount and direction by the line joining points L

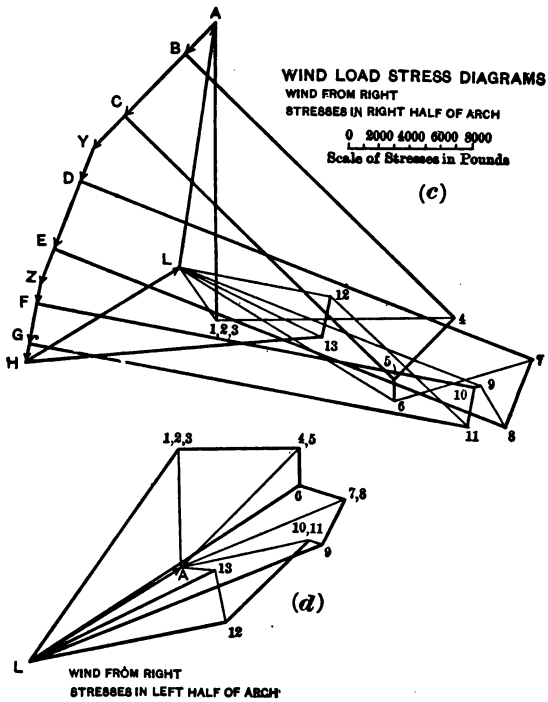


FIG. 7 (*c*) and (*d*).

and A. As there are no forces at the top-chord-panel points, the letter A is the only letter used in the member notation for top-chord members.

61. Analysis of Saw-Tooth Roof Trusses.—A roof of the saw-tooth type is usually made up of a series of similar trusses of short span, all facing with the short leg, or skylight, toward the north, or away from the direct light of the sun. The end trusses usually have one end on a

masonry wall while the intermediate trusses rest on columns as shown in Fig. 8. In some cases the end trusses also rest on columns and to resist the horizontal thrust of the wind, knee braces must be provided.

The effect of the wind blowing lengthwise of such a structure would all be taken up by the first truss, the rest of the building being sheltered from the direct force of the wind, as the wind blows practically in a

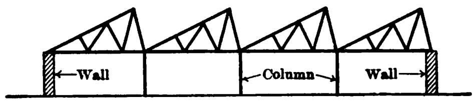


FIG. 8.

horizontal direction. The horizontal components of this force will nearly all be taken up at the wall, due to its great rigidity, very little if any going to the columns. When the wind is quartering, all of the trusses will probably be affected to some extent, although the end truss will receive the greatest pressure due to its more exposed position. The horizontal thrust of the wind forces will be transferred from the trusses, through the bottom chords and the bottom-chord bracing, to the walls and columns in proportion to their relative rigidities. As the walls are usually more rigid than the columns, the greater part of the forces will be taken at the walls.

As the spans are usually very short, and the slope very flat, it will be sufficiently accurate to determine the stresses in the members for a uniform vertical load, such as given in Table VI of Art. 53, whose effect will be equivalent to the combined action of the wind and snow. The stresses in one truss can then be determined for this load combined with the dead load, and all trusses designed alike.

We will now draw the stress diagrams for a particular case. Let the truss of Fig. 9 have a span of 25 feet, slope of longer leg 21° 48′, distance between trusses 15 feet. The loads will be taken at 1.5 pounds per sq. ft. of roof for the weight of the truss, 7.0 pounds per sq. ft. of roof for the roof covering, assuming the same kind of covering as for the Fink truss of Art. 52, and 25 pounds per sq. ft. of roof for the combined effect of snow and wind as given in Table VI of Art. 35 for a truss of 1/5 pitch (21° 48′). The total load per sq. ft. of roof will then be 33.5 pounds. As the top-chord-panel length is found to be 8.0 feet, the panel load will be $8 \times 15 \times 33.5 = 4{,}020$ pounds. Full panel

loads will be placed at each panel point, including the apex of the truss. As the joint load at the left end will not affect the stresses in the members, it will be omitted. The stress diagram as drawn for these loads is

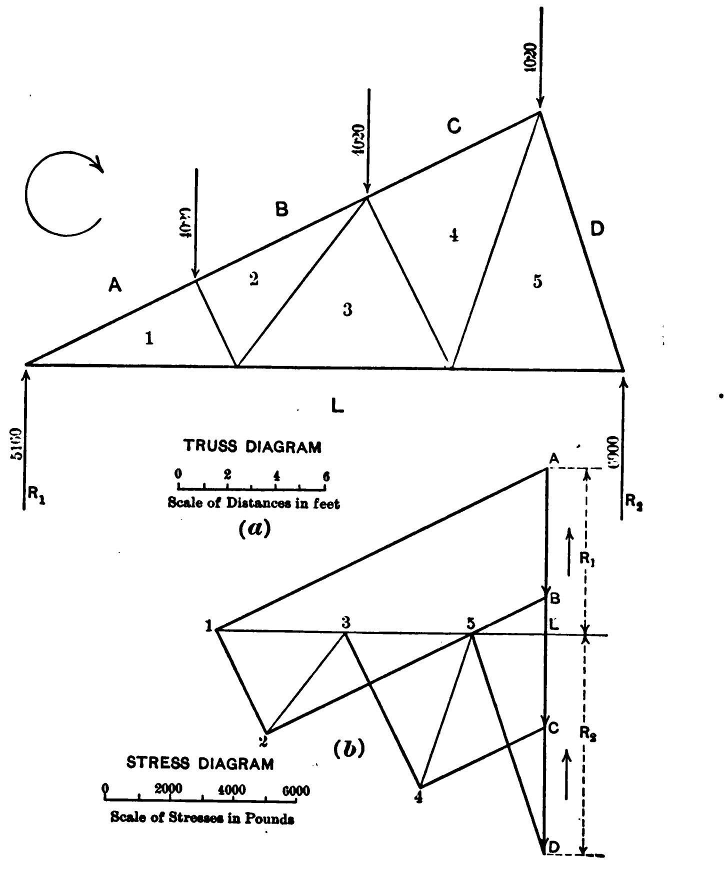


FIG. 9.

given in Fig. (*b*). Heavy lines show compressive stresses and light lines tensile stresses.

62. Trusses with Unsymmetrical Loads.—Such cases may occur when loads are suspended from the trusses, such as concentrations due to crane loads, shafting, or ceiling loads. These loads may be considered at the same time as the dead load or a separate diagram may

be drawn. The stresses as determined may then be combined with the stresses due to dead load.

To illustrate a particular case, suppose the Fink truss of Fig. 10 (*a*), span 40 feet, rise 10 feet, to carry crane loads at *M* and *N* of 20,000 pounds and 5,000 pounds respectively, and at *P*, a load of 10,000 pounds due to a heavy piece of machinery. Required the stresses in

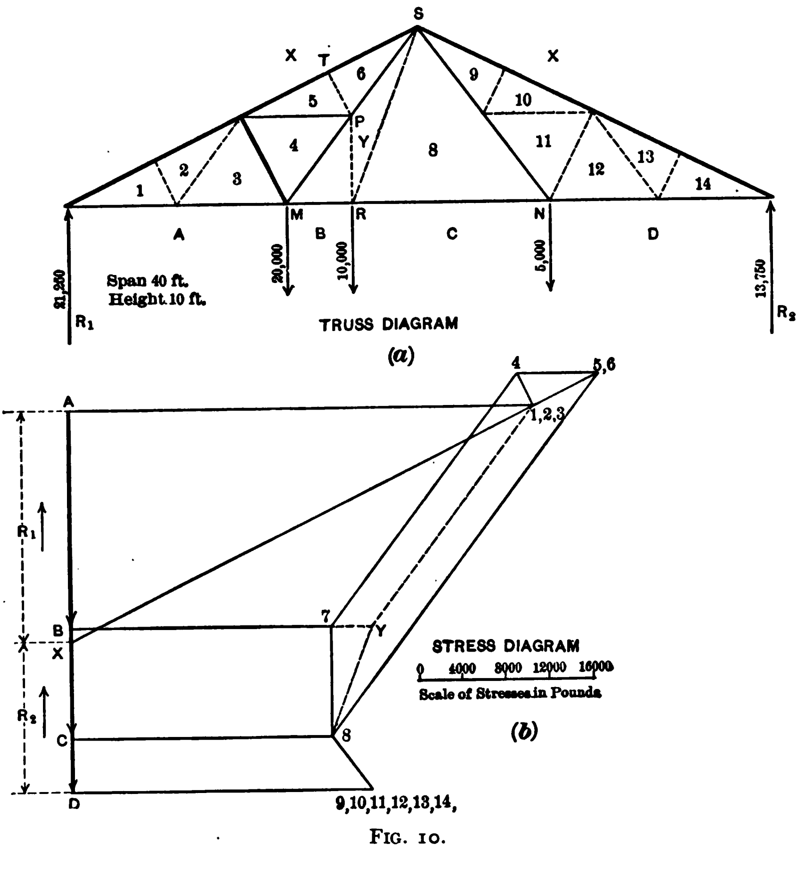


FIG. 10.

all members. To simplify the stress diagram, the load at *P* will be considered as applied at *R*, *P R* for the time being will be taken as a truss member. The load can then be treated as an external force. It can be seen that this change will not affect any of the stresses in other members or the amounts of the reactions.

To draw the stress diagram, first lay off the load line *A D*. The reactions were calculated algebraically and have the values given on the truss diagram. The easiest way to complete the stress diagram for this special case would be to begin at the right end and proceed to the left. If the diagram is begun at the left end we come at once to the second top-chord-panel point where three unknowns are in question. To pass around this joint we can make use of the special method given in Art. 58. In this case the shape of the left half of the truss will be altered in such a way that the true stress in the centre bottom-chord member, *R N*, can be obtained. This can be done by inserting the dotted member *R S* and removing all the web members on the left of the truss except *M P S*. It can be seen that the truss as altered is still a rigid frame and that the stress in *R N* will not be affected by the change. Then pass around successive joints of the bottom chord until the centre member is reached and the true stress in *R N* obtained, after which member *R S* can be removed and the original truss restored. By passing around joints *R*, *M*, and *P* again, the diagram can be completed as given in Fig. 10 (*b*). The dotted lines show the construction necessary to obtain the true stress in the centre member. The heavy lines of the truss diagram show members in compression; the light lines, those in tension, the broken lines show members whose stress is zero; and the fine dotted lines are construction lines. If loads were applied on the top chord also, it would be necessary to insert two members; *R–S*, and one from *M* to *T* (not shown). Then, by a combination of the above method and of the one given in Art 58, the unknowns can be determined.

CHAPTER IV

ANALYSIS OF BRIDGE TRUSSES FOR UNIFORM FIXED AND MOVING LOADS

SECTION I.—TYPES OF TRUSSES, LOADS AND REACTIONS

63. **Methods of Analysis.**—The particular method of analysis which will be most expeditious for a given problem depends upon the kind of loading specified and upon the form of truss. There are in general two classes or kinds of loading, (1) that in which the load is considered as uniformly distributed over all or any portion of the structure, (2) that in which the load is considered as a series of concentrated loads of given weights and spacing. The analysis of bridge trusses will therefore be treated under two general heads,

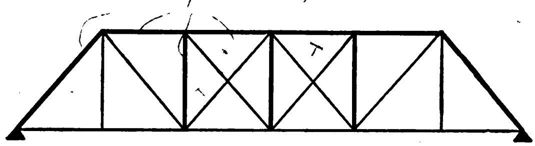


FIG. 1.—Pratt Truss.

viz., Analysis for Uniform Loads and Analysis for Concentrated Loads, no distinction being made in this respect between highway and railway bridges. Under each of these heads will be treated the various forms of trusses in common use. Analytical methods of calculation will be fully developed, but graphical methods will also be given where they are well adapted to the particular problem under discussion. The two methods are thus given side by side, an arrangement which it is believed will aid in making clear the principles involved.

64. **Modern Types of Trusses.**—The types of bridge trusses most commonly used are shown in Figs. 1–9. The Pratt truss (Fig. 1) may be considered as the standard form for spans of moderate length,

although the Warren truss (Fig. 2) is also much used. The Howe truss (Fig. 3) is the best form where it is desired to make use of

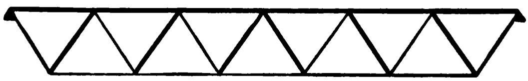


FIG. 2.—Warren Truss.

timber, steel or iron being employed generally for the vertical members only. Any one of these forms is used either as a through or a deck structure.

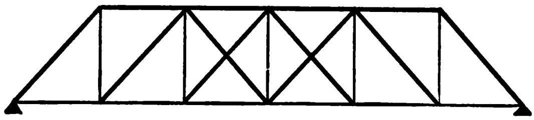


FIG. 3.—Howe Truss.

For spans greater than about 175 feet a gain in economy is secured by using a truss of variable depth, the most common form

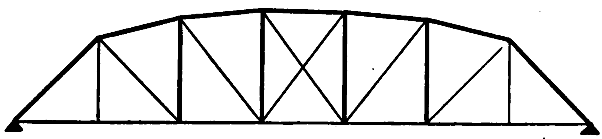


FIG. 4.—Curved-chord Pratt Truss.

being illustrated in Fig. 4. It is generally called a *curved* or *broken-chord Pratt truss.* For still longer spans it is desirable to reduce the

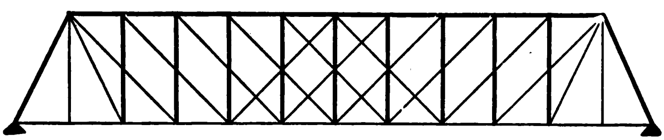


FIG. 5.—Whipple Truss.

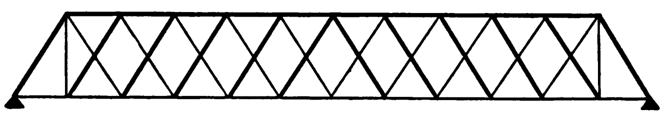


FIG. 6.—Double Warren Truss.

panel length and still retain about the same inclination of the web members. This is done in various ways, as illustrated in Figs. 5

to 9. The trusses shown in Figs. 5, 6, and 9 are known as *multiple intersection trusses*; their use was formerly quite general but in modern practice they have been displaced generally by the forms shown in Figs. 7 and 8.

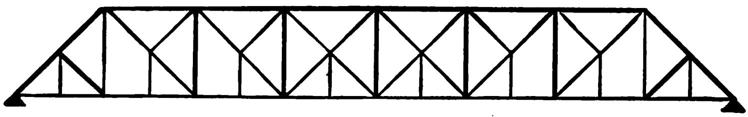


FIG. 7.—Baltimore Truss.

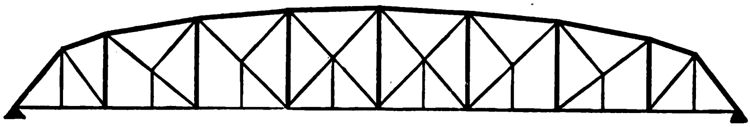


FIG. 8.—Pettit Truss.

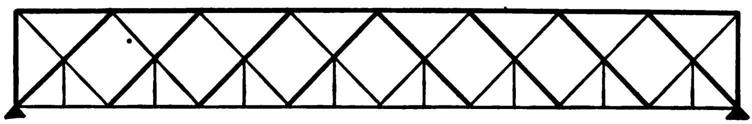


FIG. 9.—Compound Triangular Truss.

65. The Component Parts of a Truss Bridge.—The various component parts of a railroad bridge are shown in Fig. 10. They consist of: (1) The main vertical trusses; (2) the floor system; (3) the lower lateral truss; (4) the upper lateral truss; and (5) the transverse bracing. The upper and lower chords and web members of the main trusses have been defined in Chapter I. The chords of the main trusses constitute also the chords of the lateral trusses. The floor system includes all that part of the steel structure which serves to transfer the applied load to the main trusses; in the figure it is composed of longitudinal steel beams called *stringers* and transverse beams called *floor beams* which support the stringers and which are attached to the truss. Various other arrangements of beams are employed to serve this purpose as, for example, in railroad bridges supporting a ballasted track. The transverse bracing connects the two main trusses at the several panel points; at the intermediate points this bracing is commonly called *sway bracing*, while at the ends it is called *portal bracing* in through-bridges and simply

end bracing in deck-bridges. The lateral and the transverse bracing resist lateral forces, such as the wind pressure, the lateral pressure exerted by trains on a curved track, and the side thrust of moving loads due to other causes. In the following two chapters the

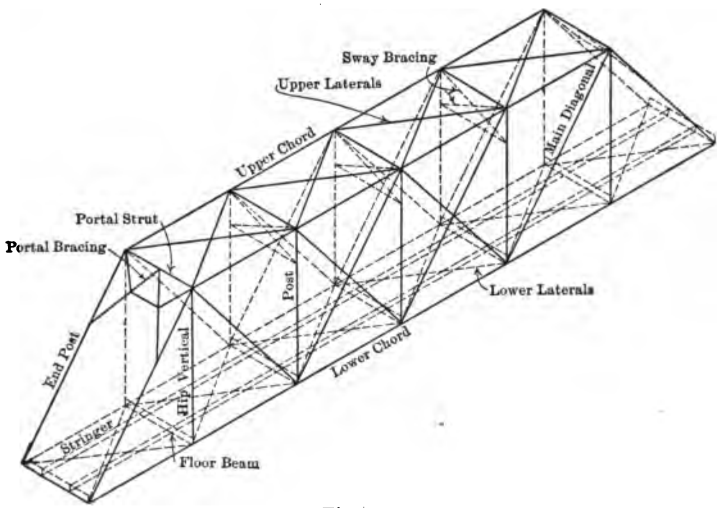


Fig. 10.

analysis of the main vertical trusses only is considered; the analysis of the various lateral and transverse trusses is fully discussed in a separate chapter.

66. The Dead Load.—The dead or fixed load consists of the weight of the supporting bridge structure itself and all other fixed weight, such as plank, ties, rails, ballast and other similar load. The weight of the bridge structure will vary with the load to be carried, the working stresses employed, the type of truss and character of details, and with the length of span; it cannot be exactly determined until the design is completed. In proceeding with a design the items making up the known dead load are first determined. Then, knowing this portion of the dead load and the live load to be carried, together with the working stresses to be employed, type of structure, etc., the weight of the steel structure (trusses and floor system) is estimated by means of an empirical formula or by comparison with weights of bridges previously designed. The total

dead load thus being approximately determined, the stresses are calculated and the design made. If the actual weight of the completed design differs essentially from the assumed weight (the allowable error depending largely upon the size of the structure), the computations must be revised.

Weights of steel-work for some of the most common types of bridges, designed under usual specifications, are given approximately by the following formulas, in which

l = span length,
w = weight per lineal foot,
p = live load per lineal foot.

Highway Bridges.—Weight of trusses and floor beams for bridges with a roadway 16 feet wide

$$w = 0.05\, l \sqrt{p} + 50. \qquad (1)$$

For the common value of p of 1,600 lbs. per ft., the formula becomes

$$w = 2\, l + 50. \qquad (2)$$

For bridges of less or greater width than 16 feet subtract or add an amount per foot equal to $0.2\, l$ for each two feet change of width. Thus, the weight of steel in a bridge 150 ft. long, 20 ft. wide, and designed for a live load of 2,000 lbs. per ft., will be approximately $0.05 \times 150 \times \sqrt{2{,}000} + 50 + (0.2 \times 150 \times 2) = 445$ lbs. per ft.

To the weight of steel must be added the weight of wooden floor, joist, etc., estimated usually at 4 lbs. per ft., B. M.

Railway Bridges.—For bridges designed for a live load consisting of two 177½-ton locomotives, followed by a uniform load of 5,000 lbs. per ft. (Cooper's $E - 50$ loading), and having open floors:

For deck plate girders

$$w = 12\tfrac{1}{2}\, l + 100. \qquad (3)$$

For through-plate girders with beams and stringers

$$w = 14\, l + 450. \qquad (4)$$

For trusses

$$w = 8\, l + 700. \qquad (5)$$

Riveted trusses of short span are likely to be somewhat heavier than given by eq. (5) and pin-connected trusses somewhat lighter. To the weight of steel must be added the weight of track, amounting to from 400 to 500 lbs. per ft. The specifications of the Am. R'y Eng. & M. of W. Ass'n require this weight to be calculated on the basis of 4½ lbs. per ft., B. M., for timber, 100 lbs. per cu. ft. for ballast and 150 lbs per lineal foot for rails and fastenings.

The formulas here given are to be considered as only roughly approximate. A more detailed discussion of the weight of different types of bridges is given in Part III.

67. The Live Load.—The live or moving load for highway bridges includes a great variety of loads, such as crowds of people, heavily loaded trucks, road rollers, electric cars, etc. For bridges not carrying street-car traffic the live load is taken as a uniform load of from 50 to 100 lbs. per square foot of roadway, or the heaviest concentrated load, due to a road roller or the like, which is likely to come upon the structure. The uniform load generally gives the maximum stresses in the main truss members, while the concentrated load usually governs the design of the floor system. For city and many country bridges electric street cars constitute a large part of the live load. For details of weights of such loads the student is referred to standard specifications.

For railroad bridges the load consists of the moving train, and acts as a series of concentrated rolling loads. The maximum load to be provided for is generally assumed to be that of two of the heaviest locomotives on the road in question, or certain standards equivalent thereto, coupled in direct position and followed by the heaviest probable train load. The loads due to the locomotive are generally treated in the calculations as concentrated loads, but the train load is usually taken as a uniformly distributed load of the same average weight per foot as the actual train. Various methods have often been employed whereby a uniformly distributed load is used in place of the actual locomotive wheel weights, such uniform load being so selected as to give approximately the same stresses as the given concentrated loads. Methods of calculation for concentrated loads and the question of "equivalent" loads are fully discussed in the next chapter.

68. Other Loads and Forces.—Besides the dead load and live load as discussed in the preceding articles, the bridge must be designed to resist other loads and forces. These include the wind pressure, the centrifugal force of trains moving on curved tracks, the tractive effort of locomotives and the friction of trains when brakes are applied, the impact or dynamic effect of the live load, temperature stresses, snow load, and sometimes other loads and forces.

The first three named forces affect chiefly the lateral trusses and are fully considered in Chapter VI.

The dynamic effect of the live load is discussed in Part III; in the analysis of stresses in this part the live load is assumed as a static load which is capable of occupying various positions on the structure. The dynamic effect due to its motion is generally taken care of by making an arbitrary increase in the specified live load or in the static live load stresses.

Changes of temperature cause changes of length in the members of a metallic structure with corresponding changes in span length. In the case of simply supported trusses this change of span length is provided for by allowing one end to move freely on its support so that no stresses are thereby produced; but where the ends are not thus free to move, as in an arch truss, the temperature changes will cause stresses which must be taken into account. These are considered in their appropriate place.

Snow load will often need consideration but the stresses resulting therefrom are determined as for dead load and require no special treatment.

69. Apex Loads and Reactions.—In the determination of the stresses in a truss all loads are assumed to be applied at joints only. The live load acts directly upon the floor system and by it is transferred to the truss at or near the joints of the lower or upper chord as the case may be; a considerable part of the dead load is similarly brought to the truss. The weight of each truss member is transferred to the joints at its two ends by the member itself, one-half being assumed as carried to each end. The entire load on the truss is thus carried at the joints and is so assumed in all truss calculations. The determination of the stresses in the floor system and in the individual members due to their own weight is a subject for separate

discussion. In the analysis of the main vertical trusses, as discussed in this and the next chapter, all loads and reactions are assumed as vertical.

The dead load is usually considered as a uniform load and assumed to be distributed among all the joints; but as the floor of a bridge constitutes a considerable part of its weight, the amount of dead load applied along the loaded chord will be considerably more than half the total. Except in the case of very large structures it is sufficiently accurate to assume two-thirds of the dead load applied at the joints of the loaded chord and one-third at the joints of the unloaded chord. For large structures it is desirable to calculate the actual distribution of weight from a preliminary design. All the live load is assumed to be applied at the joints of either the upper or the lower chord.

The panels of a truss, or the distances between joints, are usually equal; and where the load is a uniformly distributed one, each apex or joint load, except those at the ends, is equal to the load per unit length multiplied by the panel length. Where the load is not uniformly distributed, as in the case of concentrated loads, any particular joint load must be found by considering the actual loads in the panels adjacent to the joint in question.

A clear understanding of the transference and application of loads to the truss is of much importance. Fig. 11 illustrates a bridge

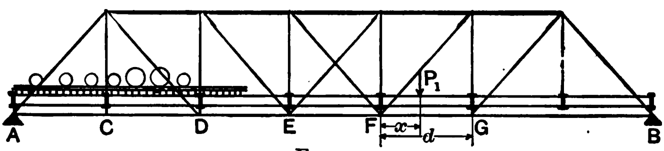


FIG. 11.

in which the floor system consists of stringers and floor beams, the latter being attached to the vertical members near the lower joints. Assume a span length of 140 feet, each panel length being 20 feet. Suppose the entire dead load is estimated at 1,500 lbs. per lineal foot, and that one-third is to be considered as applied at the joints of the upper chord, and two-thirds at the joints of the lower chord. One-half will be carried by each truss. This will give a load on each truss of 250 lbs. per ft. along the upper chord and 500 lbs. per ft.

along the lower chord. The apex load at each upper joint will then be $250 \times 20 = 5,000$ lbs., and at each lower joint (excepting the end joints A and B), $500 \times 20 = 10,000$ lbs. The load carried to the end joints A and B may be taken at one-half of the entire load for the panel, or $\frac{1}{2} \times 750 \times 20 = 7,500$ lbs. The sum of the joint loads will then equal the total load found by multiplying the load per foot by the span length. The apex loads thus determined are now to be considered applied directly at the joints in proceeding with the truss analysis; the floor system is thereafter to be entirely disregarded.

The live load reaches the stringers through the rails and ties, plank, or other supporting members; and is then transferred to the cross-beams and to the trusses. The joint loads resulting from any given single concentration, such as P_1, must first be calculated by analyzing the reactions in the floor system. In this calculation the stringers are always assumed as simple beams resting freely upon the cross-beams. The loads on the cross-beams at F and G, due to the load P_1, will therefore be equal to the end reactions of the stringer FG, considered as a simple beam. The reaction at F is equal to $P_1 \frac{d-x}{d}$, and at G it is $P_1 \frac{x}{d}$. If, further, the load P_1 is placed on the center line of the bridge, as is generally the case, the loads on the two trusses will be equal, and the joint load at F on each truss will equal $\frac{1}{2} P_1 \frac{d-x}{d}$, and at G it will equal $\frac{1}{2} P_1 \frac{x}{d}$. A uniformly distributed load equal to p per unit length will give joint loads on each truss equal to $\frac{1}{2} p\, d$. At the end joints A and B, the loads would be determined in the same way; for a uniform load each would be equal to $\frac{1}{4} p\, d$. The joint loads being thus obtained the analysis of the truss may be proceeded with.

In the detailed analysis of the truss the joint loads at A and B need not be considered, as they are each fully carried by the support underneath and cause no reaction at the other support, and hence no stresses in the truss. In what follows they will therefore be omitted unless specifically mentioned, and the reaction will be determined for the intermediate joint loads only. Thus, for the truss of Fig. 11, the dead-load reactions at A and B will each be equal

to one-half of six full joint loads, $= \frac{1}{2} [6 \times (5{,}000 + 10{,}000)] =$ 45,000 lbs. By this method of calculation the sum of the four reactions of the two trusses will be less than the total dead weight by the amount of the four half-joint loads at the ends. In designing the details at A and B the loads at these points must be considered.

70. Divisions of the Subject.—The analysis of this chapter will be arranged under the following subdivisions:

Simple beams; trusses with horizontal chords and single web systems; trusses with inclined chords and single web systems; trusses with multiple web systems; trusses with subdivided panels; skew-bridges.

Section II.—Analysis of Simple Beams; Moments and Shears

71. Before proceeding with the analysis of trusses it is desirable to consider the simple beam as a single structural unit. In the form of the rolled I-beam or plate girder, it is the common form of bridge for short spans (reaching 100 feet or more in length for railroad structures). Moreover, the truss as a whole acts as a beam and its analysis will be aided by a consideration of the solid beam. For present purposes it will be sufficient to carry the analysis only to the point of determining the bending moments and shears at any section of the beam. The intensity and distribution of the stress in the material at the section is considered as a part of the subject of design in Part III.

72. Bending Moments.—Let A B, Fig. 12, be a simple beam loaded in any manner; let q be any section and N the neutral axis at the section. Then the sum of the moments of the external forces on either side of the section, taken about the axis N, is called the *bending moment*, or simply the *moment* at N. This is balanced by the moment of the stress couple M at the section, shown in Fig. (a), whence it is said that the moment of the external forces is balanced by the moment of the internal stresses. For convenience the bending moment is called positive when it causes convexity down-

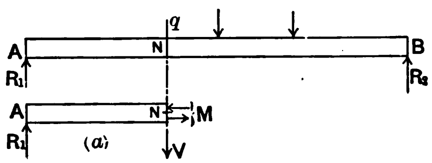


Fig. 12.

wards or produces tension in the lower fibres, and negative when the reverse. Its sign is thus seen to agree with the sign of the moment of the external forces to the left of the section, and to be the opposite of the sign of the moment of the forces to the right.

In a simple beam all downward loads cause positive bending moments at all sections.

a. Uniform Loads.—A uniform load may be *fixed*, as in the case of a dead load, or *movable*, as in the case of a live load. A movable uniform load may or may not extend entirely across a beam or bridge. In the latter case the load is uniform only so far as it goes, but the beam or bridge as a whole cannot be said to be uniformly loaded.

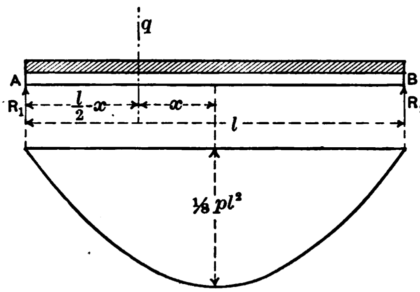


FIG. 13.

Fig. 13 represents a beam supporting a uniform load equal to p per unit length. $R_1 = \frac{1}{2}\, p\, l$. The bending moment M, at any section q, distance x from the centre, is equal to

$$R_1\left(\frac{l}{2} - x\right) - p\left(\frac{l}{2} - x\right)\, \tfrac{1}{2}\left(\frac{l}{2} - x\right).$$

This reduces to

$$M = \tfrac{1}{8}\, pl^2 - \tfrac{1}{2}\, p\, x^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

This is the equation of a parabola with vertex at the centre, at which point the ordinate is equal to $\frac{1}{8}\, p\, l^2$. Fig. 13 shows the complete moment diagram.

Equation (1) may be written in the form

$$M = \tfrac{1}{2}\, p\left(\frac{l^2}{4} - x^2\right) = \tfrac{1}{2}\, p\left(\frac{l}{2} - x\right)\left(\frac{l}{2} + x\right) \quad . \qquad (2)$$

That is, *the bending moment at any section of a beam under a uniform load equals one-half the load per foot multiplied by the product of the two segments into which the beam is divided by the section.*

In determining the maximum moments for a uniform live load such load should extend entirely across the beam; for, as already

shown, the addition of a load at any point will add to the positive bending moment at all sections. The maximum moments due to a movable uniform load are thus the same as those due to a fixed uniform load and are given by eqs. (1) and (2). These equations will enable the moments to be computed in any simple beam or plate girder for uniform loads.

b. Single Concentrated Load.—For a single concentrated load, Fig. 14, the maximum moment at any section occurs when the load is placed at that section, for a movement of the load in either direction from such position reduces the opposite abutment reaction and hence the moment.

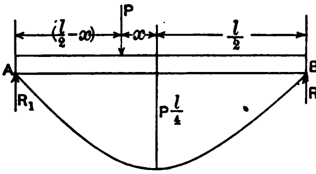


FIG. 14.

This maximum moment is given by the equation.

$$M = P\left(\frac{\frac{l}{2} + x}{l}\right)\left(\frac{l}{2} - x\right) = P\left(\frac{l}{4} - \frac{x^2}{l}\right) \quad . \quad . \quad (3)$$

This is the equation of a parabola whose ordinate is a maximum for $x = 0$, the value of this maximum ordinate being equal to $\frac{1}{4} P\, l$.

If we substitute $\frac{2P}{l}$ for p in eq. (2) we shall get eq. (3), thus showing the important principle that *the maximum bending moments due to a single moving concentrated load are the same as for twice that load when uniformly distributed over the beam.*

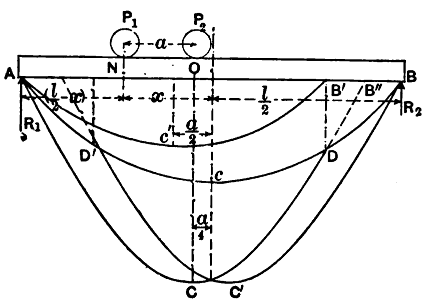


FIG. 15.

c. Two Equal Concentrated Loads.—In the case of two equal loads, a fixed distance apart, it is desirable to investigate the variation in the moment which occurs under one of the loads as they both move across the beam. In Fig. 15 the total bending moment under P_1 may readily be found by adding the moments due to P_1 and P_2 found separately. The moment due to P_1 is given by eq. (3)

and in Fig. 15 the curve $A\ c\ B$ represents this moment. The moment at N, due to P_2, is readily written out as follows:

$$M = P_2\left(\frac{\frac{l}{2} + x - a}{l}\right)\left(\frac{l}{2} - x\right) = \frac{P_2}{l}\left(\frac{l^2}{4} - x^2 - \frac{al}{2} + a\,x\right). \quad . \quad (4)$$

This is the equation of the parabola $A\ c'\ B'$, whose maximum ordinate is $\frac{P_2}{l}\left(\frac{l}{2} - \frac{a}{2}\right)^2$ when $x = \frac{a}{2}$. Adding these moments and calling each load P, the total moment under P_1 is

$$M = \frac{P}{l}\left(\frac{l^2}{2} - 2\,x^2 - \frac{al}{2} + a\,x\right). \quad . \quad . \quad . \quad . \quad (5)$$

This is represented by the curve $A\ C\ D$. Beyond D the total moment is given by eq. (3) and the curve $D\,B$.

The value of M in eq. (5) is a maximum when $x = a/4$ and is then equal to

$$M_{max.} = \frac{P}{2l}\left(l - \frac{a}{2}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is, *the maximum moment in a beam loaded with two equal moving concentrated loads occurs under one of the loads when that load is placed at a distance from the centre equal to one-fourth the distance between the loads, the other load being placed on the other side of the centre.*

The moment under P_2 is shown by the curve $A\ D'\ C'\ B$, symmetrical to $A\ C\ D\ B$.

For beams whose length is less than $2a$ it may happen that the moment at the centre point, with one of the loads placed at that point and the other off the beam, will be greater than the maximum moment given by eq. (6). By equating such centre moment ($= \frac{1}{4}\ P\ l$) with the moment of eq. (6) and solving for l, we find that the two moments are equal when $l = 1.71\ a$. For smaller values of l the centre moment, $\frac{1}{4}\ P\ l$, is therefore the maximum moment caused by the moving loads.

The curve of maximum moments for any form of loading may be found by a method similar to the above, but the subject will not be

treated further here. The succeeding chapter discusses the location of the point of maximum moment in a beam for any number of loads.

73. *Graphical Treatment of Bending Moments.*—Fig. 16 represents a beam supporting concentrated loads; Fig. (*a*) is the force

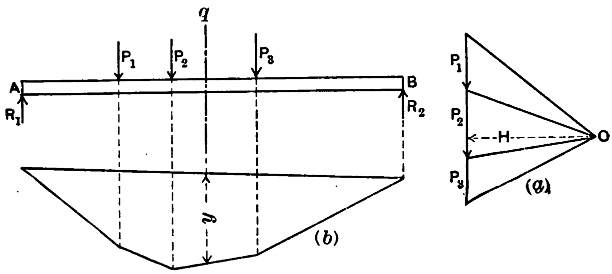


FIG. 16.

polygon and Fig. (*b*) the equilibrium polygon, constructed as explained in Chapter II. As there shown, the bending moment at any section q is given by the product of the intercept y and the pole distance H. If the load is a uniform load the equilibrium polygon will be a parabola, whose ordinates will be proportional to the bending moments. If the pole distance, H, be made unity the ordinates will be numerically equal to the moments and the equilibrium polygon becomes the ordinary moment curve as shown in Fig. 13.

For a single concentrated load, P, Fig. 17, the equilibrium polygon consists of the straight lines $A'\,C'$ and $B'\,C'$. The maxi-

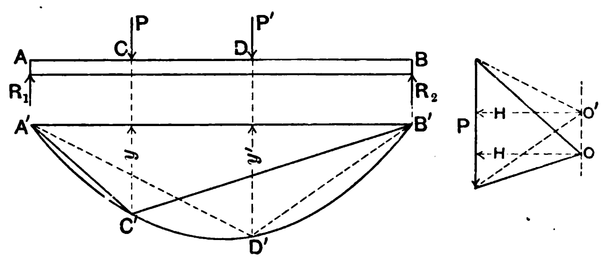


FIG. 17.

mum moment is at the load. If the load be shifted to the position D, the equilibrium polygon becomes $A'\,D'\,B'$ with ordinate y'; (using the same pole distance and keeping the closing line horizontal). The general value of the maximum ordinate y is given by

eq. (3), and as shown by that equation the locus of the point C' is the parabola $A' C' D' B'$, whose ordinates are twice as great as those of the moment parabola for the load P, if distributed uniformly over the beam.

74. Shears.—If a section be taken at any point N (Fig. 18), in a loaded beam the sum of the vertical components of the external forces upon either side of the section is called the *shear* on the section. The sign of the resultant vertical force is plus for the forces on one side of the section and minus for those on the other; but for convenience the shear is given the same sign as that of the resultant force on the left. *Positive* shear, then, is when the left-hand portion tends to move *upwards* on the right, and negative shear when the conditions are reversed. Fig. 19 illustrates in what direction positive and negative shears tend to shear a beam at a given section. From Σ vert. comp. = o it follows that the shear must be balanced by the internal force or stress in the section, that is, by the action of the portion removed upon the portion considered. This stress is called a *shearing* stress and in Fig. 18 (*a*) it is replaced by the force V.

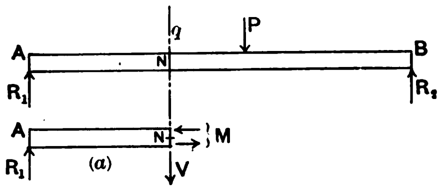


FIG. 18.

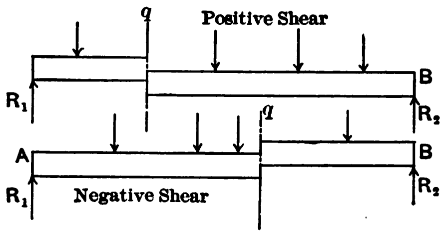


FIG. 19.

a. Uniform Loads.—The shear at any point N of a beam (Fig. 20), loaded with a fixed uniform load of p lbs. per foot, is equal to the left abutment reaction minus the load between A and N, or

$$V = R_1 - p\,x = p\left(\frac{l}{2} - x\right). \quad . \quad . \quad . \quad . \quad (7)$$

This is the equation of a straight line, $C\,D$, Fig. 22, having a maximum positive ordinate of $\frac{1}{2}\,p\,l$ when x = o, and an equal negative ordinate when $x = l$. When $x = \frac{1}{2}\,l$, V = o.

For a moving uniform load the maximum positive shear at any point N, Fig. 21, occurs when all possible loads are added to the right and when there are no loads on the left; for adding a load to the right increases the left reaction and therefore the positive shear,

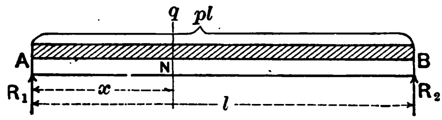


FIG. 20.

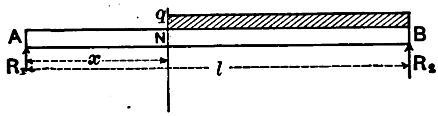


FIG. 21.

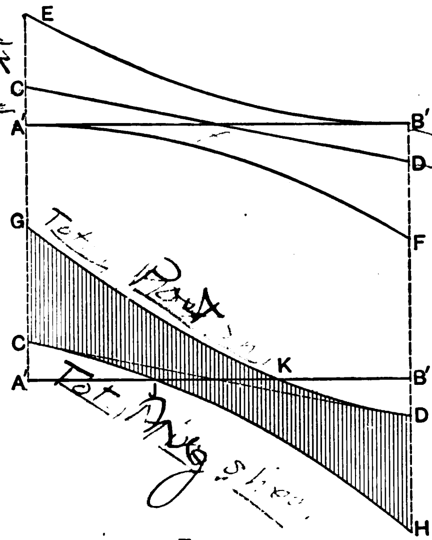


FIG. 22.

while adding loads to the left increases the left reaction by an amount less than the load itself and hence decreases the positive shear. The maximum shear at N is therefore

$$V = R_1 = p\,(l - x)\,\frac{l - x}{2l} = \frac{p}{2l}\,(l - x)^2, \quad . \quad . \quad . \qquad (8)$$

the equation of a parabola with vertex at the right end. This parabola is $B'E$, Fig. 22; the ordinate $A'E$ is equal to $\frac{1}{2}pl$. The maximum negative shear is found by loading to the *left* of the point, and is

$$V = -R_2 = -\frac{p}{2l}x^2, \quad \text{. . . . .} \quad (9)$$

the equation of the parabola $A'F$.

Combined Shears Due to Fixed and Movable Uniform Loads.—When a beam supports both a fixed and a movable uniform load (as dead and live load), the maximum positive and negative shears are found by combining the shears due to each system of loading. These combined shears are shown graphically in the lower diagram of Fig. 22 by the curves GD and CH. These curves cross the axis at J and K where the shears due to dead and live loads are equal but of opposite signs. From K to B' positive shear cannot occur and from A' to J negative shear is impossible. Between J and K both kinds of shear are possible.

To the left of J the curve CJ gives the ***minimum*** positive shears, these being less than the dead-load shears. To the right of K the curve KD likewise gives the ***minimum*** negative shears. Between J and K the upper limit of the shear is positive and the lower limit negative. In general the shears may range anywhere between the limits shown by the shaded area. The values of the shears are best found by the use of eqs. (7), (8), and (9).

In practice it is sometimes required to find maximum shears only, and sometimes both maximum and minimum shears. In either case considerations of symmetry render it necessary to analyze but one-half the beam only.

EXAMPLE.—Let it be required to calculate the dead-load shears and the maximum positive and negative live-load shears at points 5 ft. apart in a beam 60 ft. long, Fig. 23. Dead load = 400 lbs. per foot; live load = 1,200 lbs. per foot.

Dead-load Shears.—These are readily found by means of eq. (7), although it is hardly necessary to refer to a formula in this case. The resulting values are given in the table below in Column (2).

Live-load Shears.—The maximum positive shears are obtained from eq. (8) and the maximum negative shears from eq. (9). These values are given in Columns (3) and (4) of the table.

Combined Shears.—Combining the dead-load shears with the maximum positive live-

load shears gives the values of Column (5), and combining with the maximum negative live load shears gives the values of Column (6). In accordance with the explanation of Fig. 22, it should be observed that in the upper half of Columns (5) and (6) are given respectively the maximum and minimum values of the shears in the left half of the beam. At e the

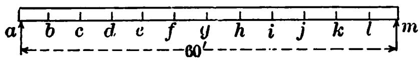


FIG. 23.

minimum is just zero; at f and g the minimum is negative; while to the left of e the minimum is positive. The values for the right half are numerically equal to but of opposite sign from those for the left half. They are given here for sake of clearness.

The results of this analysis should be carefully studied, as the general relations here shown are of much importance in truss analysis.

TABLE OF SHEARS.

Point.	Dead-load Shears.	Live-load Shears.		Combined Shears.	
		Maximum Positive.	Maximum Negative.	Dead Load and Positive Live Load.	Dead Load and Negative Live Load.
(1)	(2)	(3)	(4)	(5)	(6)
a	+ 12000	+ 36000	0	+ 48000	+ 12000
b	+ 10000	+ 30250	− 250	+ 40250	+ 9750
c	+ 8000	+ 25000	− 1000	+ 33000	+ 7000
d	+ 6000	+ 20250	− 2250	+ 26250	+ 3750
e	+ 4000	+ 16000	− 4000	+ 20000	0
f	+ 2000	+ 12250	− 6250	+ 14250	− 4250
g	0	+ 9000	− 9000	+ 9000	− 9000
h	− 2000	+ 6250	− 12250	+ 4250	− 14250
i	− 4000	+ 4000	− 16000	0	− 20000
j	− 6000	+ 2250	− 20250	− 3750	− 26250
k	− 8000	+ 1000	− 25000	− 7000	− 33000
l	− 10000	+ 250	− 30250	− 9750	− 40250
m	− 12000	0	− 36000	− 12000	− 48000

b. Concentrated Loads.—For a single load P_1, Fig. 24, the positive shear at any point N is greatest when the load is just to the right of the point, for the left reaction is then a maximum. This maximum shear is

$$V = P_1 \left(\frac{l-x}{l}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the equation of the straight line CB', Fig. 26. The ordinate $A'C = P_1$.

For two equal loads, P_1 and P_2, (Fig. 25), the maximum positive shear occurs when P_1 is just to the right of the point. The shear due to P_1 is given by eq. (10) and the line CB'. The shear due to P_2 when P_1 is at N, is equal to left abutment reaction for P_2, or

$$V = P_2 \frac{l - (x + a)}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is zero for $x = l - a$, and is equal to P_2 for $x = -a$ if that were possible; it is represented by the line $D\,E$, where $D\,B'$ and $F\,A' = a$ and $F\,E = P_2$. The total shear is the sum of the second

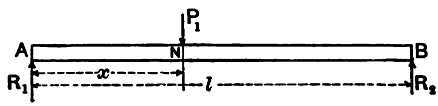


FIG. 24.

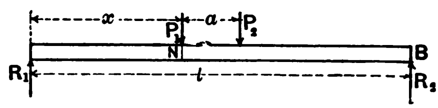


FIG. 25

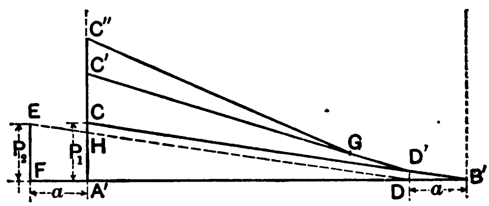


FIG. 26.

members of eqs. (10) and (11). It is found graphically by making $C'C = H\,A'$ and drawing $C'D'$ to meet $C\,B'$ in the vertical through D.

For three loads the shear diagram would be $C''\,G\,D'\,B'$, and so on, the curve approaching a parabola as the limiting case when the load is uniformly distributed.

Negative shears generally need no separate consideration, the negative shears becoming positive shears by reversing the beam.

SECTION III.—TRUSSES WITH HORIZONTAL CHORDS AND SINGLE WEB SYSTEMS

75. General Method of Determining Chord Stresses.—Fig. 27 represents a truss loaded with any given loads W_1–W_9, applied at joints only. The chord stresses are to be determined. The most direct method of procedure is by the method of sections, using for each chord member a single moment equation with centre of moments at the intersection of the opposite chord member and the web member cut by the assumed section. For example, in Fig. 27,

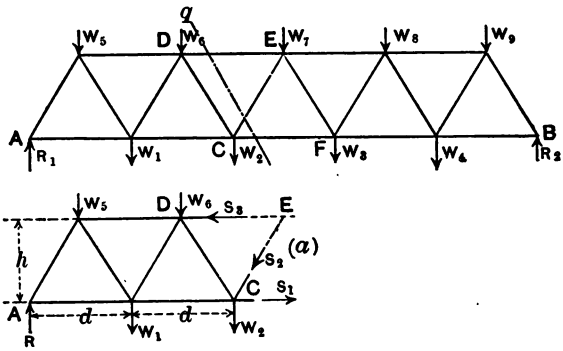


FIG. 27.

if the stress in CF is desired, pass the section q, consider the portion of the structure on the left, Fig. (a), and write Σ moments = 0 with point E as moment centre. The resulting equation will contain only one unknown quantity, S_1, it being assumed that R_1 has been previously calculated. If M_e represents the sum of the moments of the external forces we will find that $S_1 h + M_e = 0$, or $S_1 h = -M_e$. In a similar way we will find $S_3 h = -M_c$, etc. For a truss supported at the ends the value of M will always be positive, as in the case of a beam, hence the moment of the chord stress will always be negative. This requires the stress in all lower chord members to be tensile and in all upper chord members compressive. This being true, the sign of the chord stress need not be further considered, it being necessary only to find its numerical value. For this purpose it is customary to write $S_1 = \frac{M_e}{h}$ and $S_3 = \frac{M_c}{h}$, etc.

Or, in general:

The chord stress in a horizontal-chord truss with single-web system is given by the general equation

$$S = \frac{M}{h}, \qquad (1)$$

in which M = sum of the moments of the external forces on the left (or right) of the section about the opposite chord point, and h = height of truss.

In calculating M, where the panel lengths are equal, the work is simplified by using the panel length d as the unit of length and then multiplying the result so found by this length. Thus in calculating S_1 in the above example we may write $M_e = (R_1 . 2\frac{1}{2} - W_1 . 1\frac{1}{2} - W_2 . \frac{1}{2} - W_5 . 2 - W_6 . 1) \times d$. Usually all the lower joint loads are equal, and all the upper joint loads, in which case we may write $M_e = [R_1 . 2\frac{1}{2} - W_u (1\frac{1}{2} + \frac{1}{2}) - W_l (2 + 1)] \times d$, where W_u = upper joint load and W_l = lower joint load.

76. *Graphical Determination of Chord Stresses.*—In Fig. 28 construct the force polygon and equilibrium polygon for the given loads, as explained in Chapter II. $A'\,B'$ is the closing line and the reactions are given by $n\,a$ and $b\,n$ in the force polygon, $O\,n$ being parallel to the closing line. H is the pole distance. By the principles explained in Art. 40 the moment of the forces on the left (or right) of any point c, is equal to the intercept y, on the vertical through c, multiplied by H. Likewise the moment of the forces to the left of any joint C, about that joint, is given by the product of the intercept y_1, times H. But this is the moment sought in finding the stress in the chord member opposite; and, in general, the moments at the several joints needed for calculating the chord stresses are given by the corresponding intercepts in the equilibrium polygon multiplied by the pole distance. These moments divided by the height of the truss give the chord stresses. By making the pole distance equal by scale to the truss height the ordinates of the equilibrium polygon will be numerically equal to the chord stresses.

The chord stresses may also be found graphically by a force diagram of all the loads and stresses as explained in Art. 47. Such

a diagram is shown in Fig. 29. This gives not only chord stresses but web stresses as well and is advantageous for fixed loads.

77. *The Bending Moment in a Truss.*—Taken as a whole, the truss resists bending moments in the same manner as a beam, and,

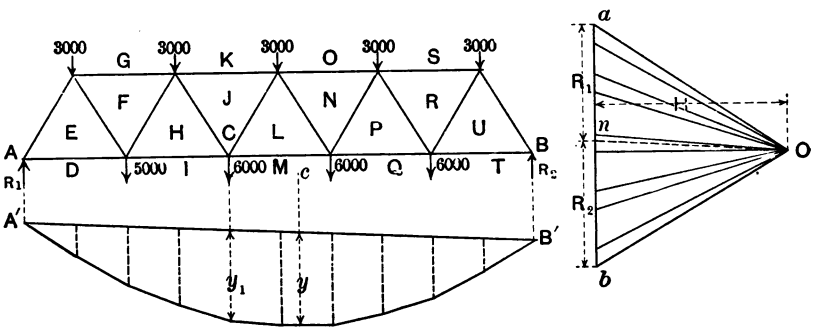


FIG. 28.

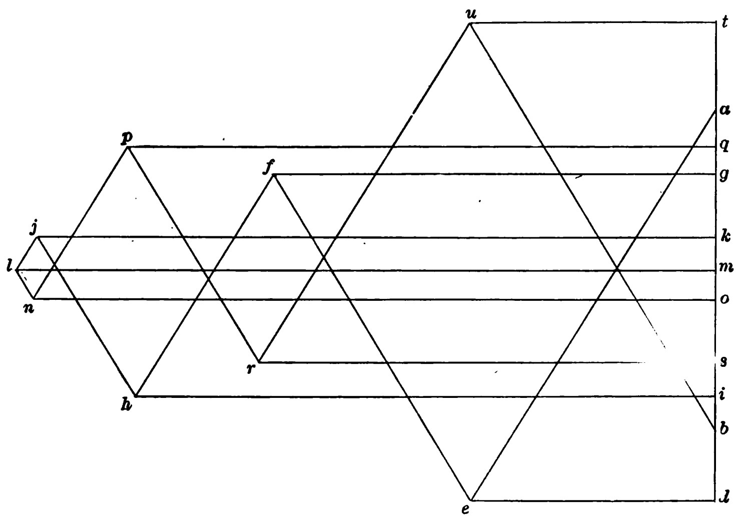


FIG. 29.

as in a beam, the bending moment may be defined as the resultant moment of the external forces to the left of any given vertical section taken about any point in the section. Referring to Fig. 27 it will be seen that the moment M_e is, by this definition, equal to the bending moment in the truss at a section through joint E, or, as generally

expressed, the bending moment at joint E. Likewise, all moments M of eq. (1) are the bending moments at the several joints opposite the chord members whose stresses are desired. Bending moments may of course be determined with reference to any section or any point between joints, as at point c in Fig. 28. These moments are not required in the analysis, but they would in fact be balanced by the moments of the stresses in the three members cut by the section in question.

78. *Bending Moments Due to a Uniform Load Applied Along One Chord.*—Let A B, Fig. 30, be any bridge in which the lower is

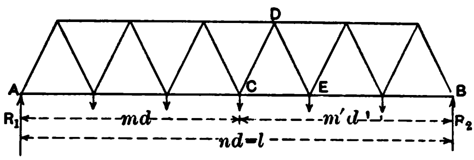


FIG. 30.

$S = \frac{M}{h}$

the loaded chord. Let d = panel length, m = number of panels on the left of any lower joint C, and m' = number of panels on the right. Suppose a uniform load of p pounds per foot per truss be applied to the bridge, producing joint loads equal to $p\,d$. There are $n - 1$ joint loads (omitting those at A and B) and therefore $R_1 = \frac{p\,d\,(n-1)}{2}$. To the left of C are $m - 1$ loads whose average distance from C is $\frac{m\,d}{2}$. The bending moment at C is therefore equal to

$$M = R_1 \times m\,d - (m-1)\,p\,d \times \frac{m\,d}{2} = \frac{p\,d\,(n-1)}{2}\,m\,d - \frac{p\,d^2}{2}\,(m-1)\,m.$$

Noting that $m + m' = n$, this becomes

$$M = \frac{p\,d^2}{2}\,(m\,m'), \quad \ldots \ldots \quad (2)$$

The stress in the chord member opposite is thus equal to

$$S = \frac{M}{h} = \frac{p\,d^2}{2\,h}\,(m\,m'). \quad \ldots \ldots \quad (3)$$

Comparing eq. (2) with eq. (2), Art. 72, it is seen that they are identical, $d\,m$ and $d\,m'$ being equal respectively to $(\frac{1}{2}\,l + x)$ and $(\frac{1}{2}\,l - x)$. That is, the bending moments at the joints of the loaded chord are the same as in a beam supporting the same uniform load.

Graphically, the moments are represented by the ordinates of the equilibrium polygon in Fig. 31. It follows from Art. 72 that the

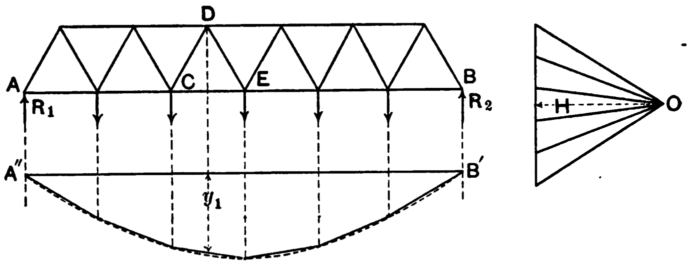


FIG. 31.

vertices of this polygon lie on the moment parabola for a uniformly loaded beam.

The bending moments at any upper joint D, not in a load vertical, cannot be obtained from eq. (2), as would be found by a detailed analysis similar to the above. In Fig. 31 this moment is represented by the ordinate y_1, drawn to the equilibrium polygon, and not by the ordinate to the parabola. The moment in the truss between joints is thus seen to be somewhat less than in a solid beam continuously loaded. The difference is in fact equal to the bending moment carried by the floor system at the section. This can be seen by computing the moments in the two cases as follows (Fig. 32). Including the load $\frac{1}{2}\,p\,d$ at A the moment for the truss can be written in the form $R_1 \times x - [\frac{1}{2}\,p\,d\,x + p\,d \times (x - d) + \frac{1}{2}\,p\,d \times (x - 2\,d)] - \frac{1}{2}\,p\,d\,z$. For the beam, Fig. (*a*), the moment is $R_1 \times x - [2p\,d \times (x - d)] - \frac{1}{2}\,p\,z^2$. The quantities within the brackets in the two expressions are equal; they

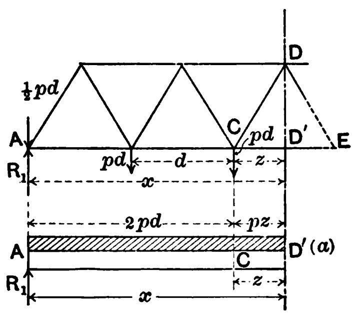


FIG. 32.

D + L = Max.
D = Min.

are the moment about D' of the portion of the load upon the length $A\,C$. The moment upon the beam is greater, therefore, than the moment upon the truss by $\frac{1}{2}\,p\,d\,z - \frac{1}{2}\,p\,z^2$. This is the bending moment at D' in the secondary member, $C\,E$, produced by the uniform load in the panel.

The equilibrium polygon shows, as an algebraic analysis would also show, that the bending moments at points intermediate between successive loaded joints, varies uniformly, that is, the moment diagram is a series of straight lines. The bending moments at intermediate points can therefore be obtained by interpolation between moments at adjacent loaded joints. Thus the moment at D is equal to a mean between the moments at C and E, etc.

79. *Maximum Moments and Chord Stresses.*—For the same reason as in the case of the beam, the maximum moments at all joints of a truss, due to a moving uniform load, will be caused when the truss is fully loaded.

80. Web Stresses.—*General Method of Determining Web Stresses.*—Let Fig. 33 represent any truss with horizontal chords and single system of bracing.

To find the stress in any web member $C\,D$, pass the section q, cutting this member and two chord members, and write Σ vertical components $= 0$ for the forces acting on the structure to the left, Fig. (*a*). Since S_1 and S_3 are horizontal, the vertical component of S_2 must balance the sum of the vertical components of the external forces. But the sum of the vertical components of the

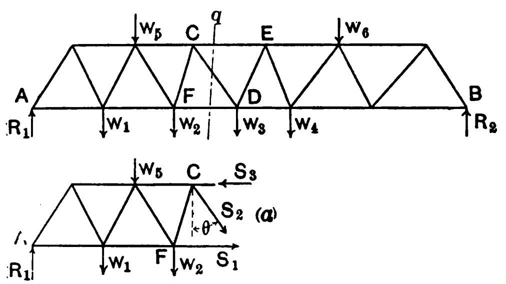


FIG. 33.

external force acting on the left of any section is called the *shear* on the section, as in the case of a beam; hence the vertical component of the stress in any web member is equal to the shear on the section cutting such web member and two chord members. The stress itself is equal to the vertical component multiplied by the secant of the angle which the member makes with the vertical. Or, in general:

$PLV = R_1 - \Sigma P$ Max S D...

$+LLV - R_1$

$-LLV - -R_2$

The web stress in a horizontal-chord truss is given by the equation

$$S = V \sec \theta, \quad \quad (1)$$

in which V = shear on the section and θ = angle which the member makes with the vertical.

Sign of the Web Stress.—In Fig. 34 (a) the shear is positive, or upwards, and the stress S_2 must act downwards or be tensile; if the shear is negative, then S_2 will act upwards and be compressive. For a member inclined in the opposite direction, as $E\,D$, Fig. 33, Fig. 34 (b) shows that positive shear would cause compressive stress,

FIG. 34.

and negative shear tensile. The nature of the stress can readily be determined by noting that the action of the portion of the structure to the right of the section upon the portion to the left must be opposed to the direction of the shear—downwards for positive shear and upwards for negative shear.

81. *Maximum and Minimum Stresses.*—Since the stress in any web member is equal to the shear multiplied by a constant, the problem of calculating web stresses in a horizontal-chord bridge reduces to one of finding maximum and minimum shears. For the same reason as given in the case of the beam, the maximum positive shear on any given section will occur when the truss is loaded as fully as possible on the right of the section and as little as possible on the left, and the maximum negative shear will occur when these conditions are reversed. Combining these maximum shears with the dead-load shears gives a series of maximum and minimum resultant shears, as in the case of a beam, which are of like signs near the ends of the truss, but of opposite signs near the centre. From this it follows that in a truss like the one shown in Fig. 33 the web members near the ends will have maximum and minimum stresses of like signs, but that near the centre the maximum and minimum stresses will be of opposite signs. That is, the members near the

ends of the truss will always be subjected to one kind of stress, while near the centre they will be subjected at certain times to one kind of stress and at other times to the opposite kind, depending on the position of the live load.

82. *Value of the Maximum Live-load Shear.—(a) Conventional Method of Calculation.*—For a uniform live load the method generally followed in calculating maximum shears is to assume all joints fully loaded on one side of the panel in question and all joints unloaded on the other side. This is evidently an impossible condition, for in order to fully load joint D, Fig. 35, for example, both adjoining

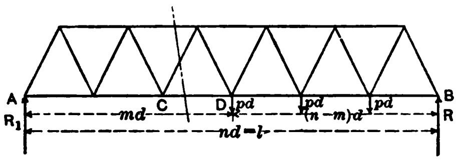


FIG. 35.

panels must be fully loaded, which condition would produce a half-joint load at C. The shears computed by this method are too great, but it is the one usually employed.

A general expression for the value of this shear for equal panels may easily be derived. Let panel $C\,D$, Fig. 35, be the mth panel from the left end, and let the maximum positive shear in this panel be required for a uniform live load of p lbs. per foot. Let n = total number of panels. Joint load = $p\,d$. All joints on the right are loaded and none on the left. The shear is equal to the left abutment reaction, $= R_1$. Taking moments about the right abutment, we have

$$R_1 \times n\,d - p\,d\,[1 + 2 + \ldots + (n - m)]\,d = 0,$$

or

$$V = R_1 = \frac{p\,d}{n}[1 + 2 + \ldots + (n - m)], \quad . \quad . \quad . \quad (5)$$

whence

$$V = \frac{p\,d}{2\,n}(n - m)(n - m + 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (5) is the simplest form to use, as it is nothing more than an abbreviated equation of moments, divided through by n.

(*b*) *Exact Method of Calculation.*—The true maximum shear in panel $C D$, Fig. 35, will evidently occur when the uniform load on the bridge floor extends some distance beyond point D into the panel $C D$. Let x equal that distance, Fig. 36. There will be full

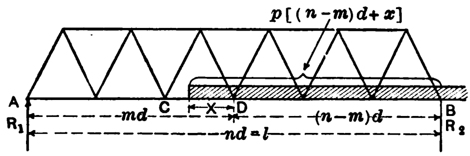


FIG. 36.

joint loads, $p\,d$, at all joints to the right of D and partial joint loads at C and D. At C the joint load will be $W_c = \dfrac{p\,x^2}{2\,d}$. The shear in panel $C\,D$ will then be equal to $R_1 - W_c$. To get R_1 most readily, consider the actual uniform load on the bridge and take moments about B. The moment of this total load will be the same as the sum of the moments of the separate joint loads. We have then $R_1 = \dfrac{p\,[(n-m)\,d + x]^2}{2\,l}$, whence

$$V' = R_1 - W_c = \frac{p\,[(n-m)\,d + x]^2}{2\,l} - \frac{p\,x^2}{2\,d}. \quad . \quad . \quad (7)$$

Differentiating this with respect to x, etc., we find that for a maximum value of V',

$$x = \frac{n-m}{n-1}\,d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting this value of x in eq. (7) and reducing, we have, for the true maximum shear,

$$V' = \frac{p\,d}{2\,(n-1)}\,(n-m)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This differs from (6) in having the factor $\dfrac{n-m}{n-1}$ in place of $\dfrac{n-m+1}{n}$, and is seen to give somewhat the smaller value. The difference is a maximum for $m = \dfrac{n+1}{2}$, or at the centre of the truss, and at that point has a value of $\dfrac{p\,d}{8} \times \dfrac{n-1}{n}$, or nearly $\dfrac{p\,d}{8}$, the same for all spans.

83. *Graphical Determination of Shears.*—In Fig. 37 the force polygon and equilibrium polygon have been drawn for the truss, as in Fig. 28. The shears can here be conveniently represented on a horizontal axis $A'\ B'$, by projecting horizontally the several forces of the load-line in the manner plainly shown in Fig. 37. This diagram then becomes

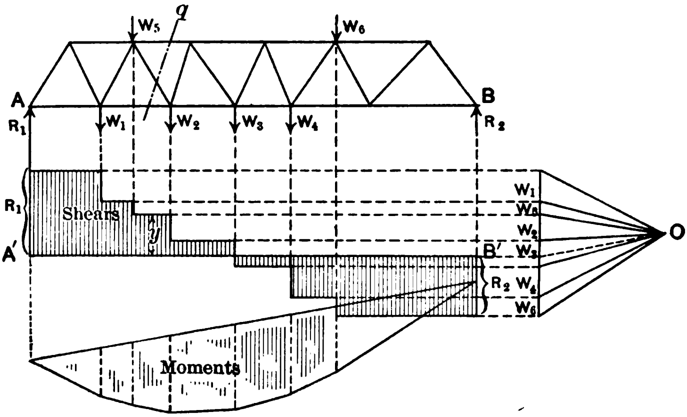


FIG. 37.

a shear diagram, the ordinates representing to scale the shears in the truss at corresponding sections. Thus the shear on section q is given by the ordinate y, etc. Note that the shear passes through zero at the point where the moment is a maximum, as proven in mechanics. Web stresses may also be found as in Fig. 29.

In practice, the shears and web stresses in trusses of this kind are so readily found by algebraic methods that graphical methods are not often employed. They are more advantageous for trusses with inclined chords. (See Art. 100.)

84. The Warren Truss.—The Warren Truss, Fig. 38, sometimes also called the triangular truss, is one in which all web members are equally inclined. It is especially well suited for deck trusses of moderate span, although also used as a through truss. It is often built as a riveted structure, and as such is commonly spoken of as a *Warren Girder* or *Lattice Girder.*

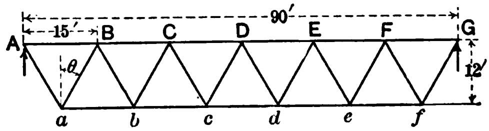


FIG. 38.

Assume the following data: Span length = 90 ft.; height of truss = 12 ft.; dead load per foot per truss = 600 lbs.; live load per foot per truss = 1,600 lbs. The dead joint load = W = 600 × 15 = 9,000 lbs.; live joint load = P = 1,600 × 15 = 24,000 lbs. We will first assume that the entire dead load is applied at the upper chord joints.

Chord Stresses; *Dead Load.*—Left reaction, $R_1 = \frac{5 \times 9{,}000}{2} =$ 22,500 lbs. Then applying the method explained in Art. 75, the lower chord stresses may be written out as follows:

$a\,b$ (moment center at B) = 22,500 × 15/12 = 28,125 lbs.

In a similar manner

$b\,c$ = (22,500 × 2 − 9,000 × 1) 15/12 = 45,000 lbs.
$c\,d$ = [22,500 × 3 − 9,000 × (1 + 2)] 15/12 = 50,625 lbs.

The upper chord stresses are as follows:

$A\,B$ = 22,500 × ½ × 15/12 = 14,060 lbs.
$B\,C$ = (22,500 × 1½ − 9,000 × ½) 15/12 ≐ 36,560 lbs.
$C\,D$ = [22,500 × 2½ − 9,000 × (½ + 1½)] 15/12 = 47,810 lbs.

All upper chord stresses are compressive and all lower chord stresses tensile.

A somewhat briefer method of calculating the chord stresses is by the use of eq. (3), Art. 78, but it should be carefully noted that this formula will apply here to the *lower* chord members only. In this case $\frac{p\,d^2}{2\,h} = \frac{600 \times 15^2}{2 \times 12} = 5{,}625$. Then stress in $a\,b$ = 5,625 × (1 × 5) = 28,125 lbs.; stress in $b\,c$ = 5,625 (2 × 4) = 45,000 lbs.; etc.

For the moments at the lower joints and stresses in the upper chord members we may use the method of interpolation as explained in Art. 78. Thus the moment at a will be one-half the moment at B; the moment at b will be one-half the sum of the moments at B and C, etc. The corresponding chord stresses will then follow the same rule and we have

$$A\,B = \frac{28{,}125}{2} = 14{,}060 \text{ lbs.}$$

$$B\,C = \frac{28{,}125 + 45{,}000}{2} = 36{,}560 \text{ lbs.}$$

etc., etc.

Chord Stresses; Live Load.—These may be found in the same way as the dead-load stresses, or they may be obtained by multiplying the dead-load stresses by the ratio of live to dead load = 1,600/600. The stresses are as follows:

$a\,b$ = 75,000 lbs.	$A\,B$ = 37,500 lbs.
$b\,c$ = 120,000 lbs.	$B\,C$ = 97,500 lbs.
$c\,d$ = 135,000 lbs.	$C\,D$ = 127,500 lbs.

Web Stresses; Dead Load.—(Fig. 39.) Applying the method of Art. 80, we will first find the shears on the several sections, such as q, cutting a web member and two chord members. There being loads at the upper joints only it is evident that the shear on the section cutting

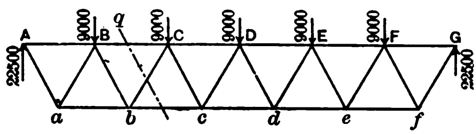


FIG. 39.

$B\,b$ is the same as on the section cutting $b\,C$; also the shear on the section cutting $A\,a$ is the same as that on the section cutting $a\,B$, etc. In other words, the shear on any section cutting through a given panel of the loaded chord is constant. The shears in these several panels are as follows:

Shear in panel $A\,B = R_1 = +\ 22{,}500$.
Shear in panel $B\,C = R_1 - W = 22{,}500 - 9{,}000 = +\ 13{,}500$.
Shear in panel $C\,D = R_1 - 2\,W = 22{,}500 - 18{,}000 = +\ 4{,}500$.

The shears on the right of the center are the same, but of opposite sign.

The web stresses are numerically equal to the shears multiplied by sec θ, the value of which $= \dfrac{\sqrt{12^2 + 7.5^2}}{12} = 1.179$. On the left of the centre, where the shear is positive, all members inclining upwards towards the abutment ($A\,a$, $B\,b$, and $C\,c$), are in tension, and members inclining downwards towards the abutment ($a\,B$, $b\,C$, and $c\,D$), are in compression. On the right the signs are reversed. The values of the stresses are given in Fig. 40(a) and are marked "D."

Web Stresses; Live Load.—The conventional method of calculation will first be used (Art. 82). For the maximum positive shear in panel $A\,B$ the entire bridge is loaded, and the shear $= R_1 = 5\,P/2 = 60{,}000$ lbs. For panel $B\,C$, joints C to F inclusive are loaded. The shear is given by eq. (5). In this case $\dfrac{p\,d}{n} = \dfrac{1{,}600 \times 15}{6} = 4{,}000$, and we have

$$\text{Shear in panel } B\,C = 4{,}000\,(1 + 2 + 3 + 4) = 40{,}000.$$

Likewise

$$\text{Shear in panel } C\,D = 4{,}000\,(1 + 2 + 3) = 24{,}000.$$
$$\text{Shear in panel } D\,E = 4{,}000\,(1 + 2) = 12{,}000.$$
$$\text{Shear in panel } E\,F = 4{,}000 \times 1 = 4{,}000.$$

The maximum negative shears will occur with the truss loaded on the left of the panel in question. They will be of the same numerical value as the above positive shears but reversed in order.

The following table gives the shears and corresponding stresses in one-half the truss.

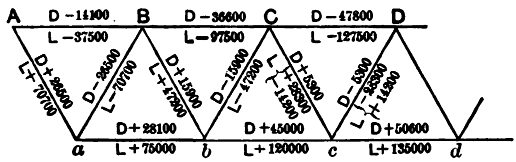
LIVE-LOAD SHEARS AND WEB STRESSES.

Member.	Positive Shears.	Stresses.	Negative Shears.	Stresses.
$A\,a$	60,000	+ 70,700	0	
$a\,B$		− 70,700		
$B\,b$	40,000	+ 47,200	4,000	− 4,700
$b\,C$		− 47,200		+ 4,700
$C\,c$	24,000	+ 28,300	12,000	− 14,200
$c\,D$		− 28,300		+ 14,200

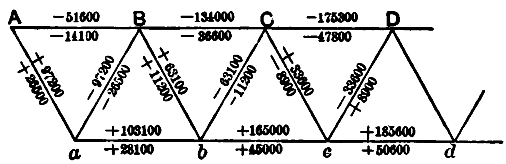
Combined Dead- and Live-load Stresses.—In Fig. 40 are brought together in a convenient form all the dead- and live-load stresses just found. For the web members the maximum live-load stresses of both kinds are given. In using these results in designing, some specifications require only a knowledge of the maximum stress, tension or compression, to which the member will be subjected; other specifications require also the calculation of the minimum stress even though it is of the same sign as the maximum. If only the maximum be required, the small live-load stresses in members $B\,b$ and $b\,C$, due to negative shears, are not needed; but the stresses of 14,200 lbs. in $C\,c$ and $c\,D$

are needed, as they are larger than the dead-load stresses, thus giving a reversal of stress in these members. Hence, in general, where only maximum stresses are desired the live-load negative shears need be found only in those panels where they exceed the dead-load positive shears.

If the minimum stresses are required, as well as the maximum, then the web stresses due to negative shears will be needed throughout, as the minimum stress in $b\,C$, for example, will be equal to 15,900 − 4,700 = 11,200 lbs. Fig. 40 (*b*) gives the maximum and minimum stress in each member, all the results being obtained from Fig. (*a*).



(*a*) Dead- and Live-Load Stresses.



(*b*) Maximum and Minimum Stresses.

FIG. 40.

The minimum stress in all chord members is the dead-load stress; the maximum is the sum of the dead- and live-load stresses. In $C\,c$ and $c\,D$ the stress which is here called the minimum is really the maximum stress of opposite sign from the one which is numerically the greater and which is usually called the maximum. The stresses in these members may be anything between these limits and may therefore be zero. This is numerically the minimum value, but the information needed in designing is the greatest range of stress to which the member can be subjected and hence, where the stress can reverse in sign, it is the maximum values of the two opposite kinds of stress that should be calculated. The minimum stresses in $A\,a$ and $a\,B$ are the dead-load stresses.

The maximum and minimum web stresses may also be found conveniently by first combining the shears and then calculating the resultant stresses at once, as in the following table:

COMBINED SHEARS AND WEB STRESSES.

Member.	Dead-Load and Positive Live-Load Shears.	Stresses (Maximum).	Dead-Load and Negative Live-Load Shears.	Stresses (Minimum).
A a	+ 82,500	+ 97,200	+ 22,500	+ 26,500
a B		− 97,200		− 26,500
B b	+ 53,500	+ 63,100	+ 9,500	+ 11,200
b C		− 63,100		− 11,200
C c	+ 28,500	+ 33,600	− 7,500	− 8,900
c D		− 33,600		+ 8,900

True Maximum Live-Load Shears.—The shears resulting from the application of eq. (9), Art. 82, are as follows:

Panel $AB = 60{,}000$ Panel $BC = 38{,}400$ Panel $CD = 21{,}600$.
Panel $DE = 9{,}600$. Panel $EF = 2{,}400$.

Comparing these with the live-load shears previously found it will be seen that they are materially less. The members most affected by the errors of the approximate method are those near the centre of the bridge, a result not altogether undesirable.

Effect of Applying a Part of the Dead Load at the Lower Chord Joints.—If one-third the dead load be assumed as applied along the lower chord, then each upper joint load may be taken as equal to $\frac{2}{3}W = 6{,}000$ lbs., and each lower joint load as equal to $\frac{1}{3}W = 3{,}000$ lbs., the actual loads at a and f being very nearly as great as at the other joints.

This distribution will give slightly different reactions from those already found, each being increased by $\frac{1}{6}W$, which amount has in effect been transferred from the end joints to the adjacent lower joints. The shears on the sections cutting the several members will all be different from those previously found. Those on sections cutting members $A\,a$, $B\,b$, and $C\,c$ will be increased by $\frac{1}{6}W$ and those on sections cutting $a\,B$, $b\,C$, and $c\,D$ will be decreased by $\frac{1}{6}W$. The lower chord stresses will be unchanged, but the upper chord stresses will be slightly modified. They should be found by applying the method of moments in detail. Thus for $C\,D$ (moment centre at c)

$$\text{Stress} = [R_1 \times 2\tfrac{1}{2} - \tfrac{2}{3} W (1\tfrac{1}{2} + \tfrac{1}{2}) - \tfrac{1}{3} W (2 + 1)]\ 15/12, \text{etc.}$$

Equation (3), Art. 78, will not apply in this case except for that part of the stress due to the loads on the upper chord joints. The live-load chord stresses cannot in this case be calculated from the dead-load stresses.

85. The Warren Truss with Verticals.—When the Warren truss is used for spans of considerable length it is desirable to subdivide the panels of the loaded chord by the use of additional vertical members (Fig. 41), thus shortening the length of the longitudinal floor members. These verticals carry the joint loads applied at the loaded chord joints b, d, f, etc. In the deck-bridge they are compression members, in the through-bridge tension members.

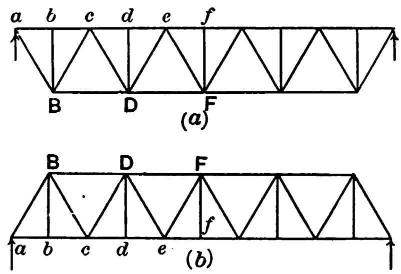


FIG. 41.

The stresses in all diagonals are determined from the shears, as in the truss just analyzed, and the chord stresses from moments. On account of the supporting joints at b, d, f, etc., the stresses in the diagonals of successive panels will be different. The chord stresses $a\,b$ and $b\,c$ are equal; also $c\,d$ and $d\,e$, etc.

EXAMPLE.—(Fig. 42). Span length = 150 ft.; height of truss = 25 ft.; dead load per foot per truss = 900 lbs.; live load per foot per truss = 1,600 lbs. Assume one-third of dead load applied at upper chord joints. The dead joint loads may be determined by assuming the load applied along the lower chord at 600 lbs. per foot, and that along the upper chord at 300 lbs. The lower joint loads, W_l, will then be 600 × 15 = 9,000 lbs., and the upper joint loads, W_u, will be 300 × 30 = 9,000 lbs. The load at B may be taken at 9,000 lbs. also. The joint load at a is neglected. $R_1 = \frac{1}{2}(9 \times 9{,}000 + 5 \times 9{,}000) = 63{,}000$. Each live joint load $= P = 1{,}600 \times 15 = 24{,}000$ lbs. The solution will be carried only far enough to illustrate the method, the complete analysis being left to the student.

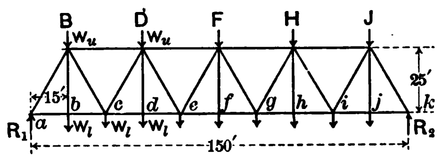


FIG. 42.

Dead-Load Stresses.—For chord stresses we have, for example,

$$c\,d \text{ and } d\,e = M_D/25$$
$$= [63{,}000 \times 3 - 9{,}000\ (2 + 2 + 1)]\frac{15}{25} = 86{,}400 \text{ lbs.}$$

$$D F = M_e / 25$$
$$= [63{,}000 \times 4 - 9{,}000 (3 + 3 + 2 + 1 + 1)] \frac{15}{25} = 97{,}200 \text{ lbs.}$$

For web stresses, sec. $\theta = \sqrt{15^2 + 25^2}/25 = 1.17$.

$B c =$ shear in panel $b c \times 1.17$
$= (63{,}000 - 9{,}000 \times 2)\ 1.17 = 52{,}500$ lbs. tension.

$c D =$ shear in panel $c d \times 1.17$
$= (63{,}000 - 9{,}000 \times 3)\ 1.17 = 41{,}800$ lbs. compression.

Live-Load Stresses.—The lower chord joints are loaded with 24,000 lbs. each. The moments may be obtained as for dead load, or by the parabolic formula. Fig. 43 shows the moment polygon, the vertices of which lie on a parabola, since the joint loads are produced by a uniform load. It will be seen that all moment centres for both top and bottom chords are in load verticals and all the desired moments are therefore given by the ordinates to the parabola. We then have by equation (3) Art. 78, $M_C = p\, d^2/2\ (mm') = 180{,}000\ (2 \times 8) = 2{,}880{,}000$ ft.-lbs.; $M_D = 180{,}000\ (3 \times 7) = 3{,}780{,}000$ ft.-lbs., etc.

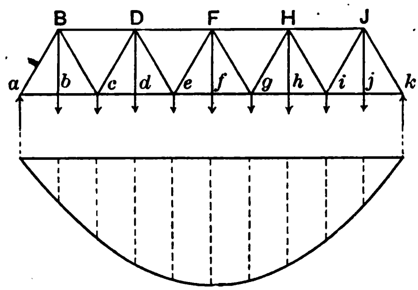


FIG. 43.

The maximum positive shear in panel $b c$ requires all joints loaded except b. The shear is $R_1 = P/10\ (1 + 2 + \ldots + 8) = 2{,}400 \times 36 = 86{,}400$ lbs. Likewise the maximum shear in panel $c d = P/10\ (1 + 2 + \ldots + 7) = 2{,}400 \times 28 = 67{,}200$ lbs., etc.

86. The Pratt Truss.—This form of truss, shown in Fig. 44 as a through-bridge, is the most common type of truss for spans from 100 to 250 feet in length. For the shorter spans it is frequently built as a riveted bridge, but for long spans is made pin-connected. The members *B-b* and *H h* are called *hip verticals*, or *end suspenders;* they are

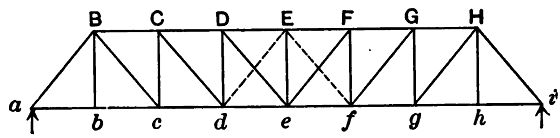


FIG. 44.

tension members and support only the joint loads at *b* and *h*. All other verticals are compression members. All diagonals, except the end posts *a B* and *H i*, are tension members. The dotted diagonals, *d E* and *E f*, are called *counters;* they are used in those panels where the maximum and minimum shears are of opposite signs. By their use

the members De and eF are not subjected to compressive stresses when the shears in the respective panels change sign, but instead they simply drop out of action (the stress becomes zero), and the counters carry the shear in tension. Sometimes counters are not used, the main diagonals being designed for alternating stresses in those panels where counters would otherwise be needed.

We will assume the following data for the truss shown in Fig. 44: Span = 128 ft.; height = 20 ft.; dead load = 450 lbs. per foot per truss; live load = 1,000 lbs. per foot per truss; d = 16 ft.; W = 16 × 450 = 7,200; P = 16 × 1,000 = 16,000. All the dead load will at first be assumed applied at the lower chord joints, and afterwards a portion will be transferred to the upper joints and the stresses corrected.

Chord Stresses.—As the chord stresses are a maximum under full load, the counters are not in action and should therefore be omitted from consideration (see Fig. 45). The stress in any chord member is

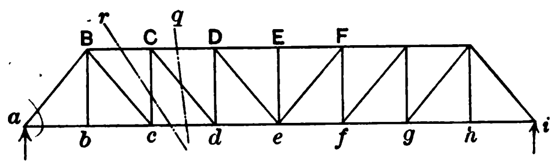


FIG. 45.

readily found by the methods already explained. Thus for member de the moment centre is D and we have as before: Stress = $[R_1 \times 3 - W(2+1)]\ 16/20$. For member CD the moment centre is d and the stress will be numerically the same as found for de. Likewise the stresses in BC and cd are numerically equal, also those in ab and bc.

Equation (3), Art. 78, can be used conveniently in this truss. In the case of dead load $\dfrac{pd^2}{2h} = \dfrac{450 \times 16^2}{2 \times 20} = 2,880$. Noting carefully the moment centre for each member, the following results will readily be obtained:

$$
\begin{aligned}
ab \text{ and } bc &= 2,880 \times 1 \times 7 = 20,160.\\
BC \text{ and } cd &= 2,880 \times 2 \times 6 = 34,560.\\
CD \text{ and } de &= 2,880 \times 3 \times 5 = 43,200.\\
DE \quad &= 2,880 \times 4 \times 4 = 46,080.
\end{aligned}
$$

The live-load stresses can be found from the dead-load stresses as in Art. 84.

Web Stresses.—The same general methods apply here as in the Warren truss. Thus the vertical component of the stress in $C\,d$ = shear on section q; vertical component of stress in $C\,c$ = shear on section r, etc. There being no load at C, the shear on q = shear on r, hence the stress in $C\,c$ = vertical component in $C\,d$. Likewise stress in $D\,d$ = vertical component in $D\,e$, etc. The stress in $B\,b$ is equal to the load at b and has a maximum value of $P + W$ and a minimum value of W.

The dead-load shears are as follows:

$$\text{Shear in panel } a\,b = R_1 = \frac{7 \times 7{,}200}{2} = +\ 25{,}200.$$
$$\text{Shear in panel } b\,c = 25{,}200 - 7{,}200 = +\ 18{,}000.$$
$$\text{Shear in panel } c\,d = 25{,}200 - 2 \times 7{,}200 = +\ 10{,}800.$$
$$\text{Shear in panel } d\,e = 25{,}200 - 3 \times 7{,}200 = +\ 3{,}600.$$

The maximum positive live-load shears, calculated by the conventional method, are:

$$\text{Shear in panel } a\,b = R_1 = \frac{7 \times 16{,}000}{2} = 56{,}000.$$
$$\text{Shear in panel } b\,c = \frac{16{,}000}{8}(1 + 2 + 3 + 4 + 5 + 6) = 42{,}000.$$
$$\text{Shear in panel } c\,d = 2{,}000\,(1 + 2 + 3 + 4 + 5) = 30{,}000.$$
$$\text{Shear in panel } d\,e = 2{,}000\,(1 + 2 + 3 + 4) = 20{,}000.$$
$$\text{Shear in panel } e\,f = 2{,}000\,(1 + 2 + 3) = 12{,}000.$$
$$\text{Shear in panel } f\,g = 2{,}000\,(1 + 2) = 6{,}000.$$
$$\text{Shear in panel } g\,h = 2{,}000 \times 1 = 2{,}000.$$

The maximum negative shears will be equal to these, but reversed in order.

Combining the dead-load with the maximum positive and negative live-load shears for the left half of the truss, we have the following:

Panel.	$a\,b$	$b\,c$	$c\,d$	$d\,e$
Dead-load shear	+ 25,200	+ 18,000	+ 10,800	+ 3,600
Positive live-load shear	+ 56,000	+ 42,000	+ 30,000	+ 20,000
Negative live-load shear	0	− 2,000	− 6,000	− 12,000
Maximum shears (dead load + pos. live load)	+ 81,200	+ 60,000	+ 40,800	+ 23,600
Maximum shears (dead load + neg. live load)	+ 25,200	+ 16,000	+ 4,800	− 8,400

It is to be noted that in all panels but *d e* the maximum and minimum shears are both positive. These shears, multiplied by sec θ, give therefore the maximum and minimum stresses in the respective diagonals. In the panel *d e* the maximum shear is positive and gives the maximum stress in the main diagonal *D e;* the minimum shear is negative and gives the maximum stress in the counter *d E.* Both of these stresses are tensile. The least stress in each of these diagonals is zero, which occurs whenever the shear is of such sign that it is carried by the other member.

Referring now to the vertical compression members, it will be seen that the stress in *C c* (Fig. 45), is always equal to the shear on section *r*, which is the same as the shear in panel *c d*, there being no load at *C*. The maximum stress in *C c* is therefore 40,800 lbs. and the minimum is 4,800 lbs. The stress in *D d* is likewise equal to the shear in panel *d e*, if the counter is not in action. This is the case when the shear is a positive maximum and hence the maximum stress in *D d* = 23,600 lbs. When the shear in panel *d e* is negative, then the counter *d E* is acting and the members constituting the truss are as shown in Fig. 46.

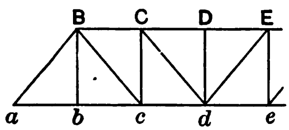


FIG. 46.

The stress in *D d* under these conditions is not controlled by the shear, but is found by considering the forces acting on joint *D*. There being no load at *D*, the stress in *D d* will be zero and this is the minimum value desired. The stress in *E e* must also be determined with careful reference to the members which are in action under any given loading. The maximum stress will occur when *E f* or *d E* receives its maximum. When *E f* is stressed, *D e* is also in action (Fig. 47), and the stress in *E e* will equal the shear on section *s*. This will be equal to − 3,600 lbs. dead-load shear + 12,000 lbs., live-load shear = 8,400 lbs., which is the same as the vertical component in *E f* (or *dE*). Note that the dead-load portion of the maximum stress in *E e* is tension, that is, the dead load acts to reduce the live-load compression. This action is similar to that which occurs in the counters. The minimum stress in *Ee* is equal to zero, as in the case of *D d.*

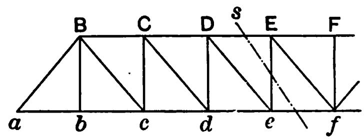
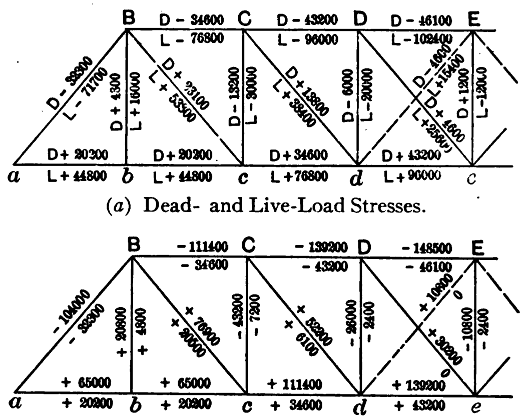


FIG. 47.

Correction for the Loads Applied at the Upper Joints.—If one-third of a dead joint load be transferred from each lower chord joint to the upper chord joint above, the only members whose stresses will be affected will be the verticals, for none of the bending moments will be changed, nor the shears on the vertical sections cutting the diagonal members. The shears on the sections cutting the verticals $C\,c$ and $D\,d$ will be increased by $\frac{1}{3}\,W$, hence the dead-load compression in these members will be increased by $\frac{1}{3}\,W$. In member $E\,e$, Fig. 47,



(*a*) Dead- and Live-Load Stresses.

(*b*) Maximum and Minimum Stresses.

FIG. 48.

the negative dead-load shear on section *s* will be decreased by $\frac{1}{3}\,W$; therefore the dead-load tension will be decreased by this amount, and the combined dead- and live-load stress will be increased by $\frac{1}{3}\,W$. The minimum stress in $D\,d$, Fig. 46, and likewise in $E\,e$, will now be $\frac{1}{3}\,W$. Finally, the dead-load stress in $B\,b$ will be decreased by $\frac{1}{3}\,W$. These results lead to the general statement that the transferring of a part of the panel load to each upper chord joint increases the compression or decreases the tension in each vertical by the amount so transferred.

Summary of Stresses.—In Fig. 48 are given the results of this analysis, with stresses in the verticals corrected for a load of $\frac{1}{3}\,W$, = 2,400 lbs., applied at each upper joint. Fig. (*a*) gives the dead- and live-load stresses which act together to produce the maximum stresses in the various members; Fig. (*b*) gives maximum and minimum stresses.

Note that the maximum of Fig. (*b*) can be obtained by adding the dead- and live-load stresses of Fig. (*a*), but that the data of Fig. (*a*) are not sufficient for calculating the minimum web stresses of Fig. (*b*).

87. The Deck Pratt Truss.—As a deck-bridge the Pratt truss may be supported either at the bottom chord, as in Fig. 49, or at the top

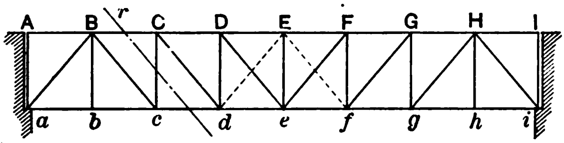


FIG. 49.

chord, as in Fig. 50. The former method is usually employed where there are several spans, and the latter where there is but a single span. In Fig. 49 members *A a* and *A B* are not members of the truss proper.

In both forms the stresses in chords and diagonals are the same as in the through-bridge, since all corresponding moments and shears are the same. In Fig. 50 the end diagonals *A b* and *I h* are tension mem-

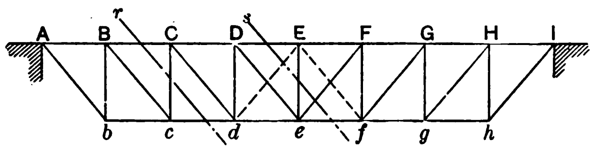


FIG. 50.

bers. The stresses in the verticals are greater than in the through-bridge. Thus the stress in *C c* is equal to the shear on section *r*, as before, but this shear is a maximum when the bridge is loaded up to joint *C* instead of to joint *d* as in Fig. 45. The live-load stress in *C c* is then equal to the shear in panel *B C*, which is the same as the shear in panel *b c* of the through-bridge. The dead-load stresses are increased by whatever increase there may be in the proportion of the joint load assumed to be applied at the upper joints, this increase usually amounting to ⅓ *W*. In Fig. 49 member *B b* receives a total stress of only ⅓ *W*, the load at *b*. In Fig. 50 this member carries shear exactly as *C c*, and its stress is determined in a similar manner.

In calculating the stress in the middle vertical *E e*, it is necessary first to ascertain which of the diagonals in panel *E F* is in action. To

do this, calculate the *total* shear in this panel with the bridge loaded up to joint E. If this shear is positive, then $E f$ will be in action; if negative, then $e F$. In case $E f$ is acting, then the stress in $E e$ will equal the shear on section s; but if $e F$ is acting, the stress in $E e$ will equal the total load at E. As a matter of fact, if the maximum positive shear on section s is greater than the total load at E, it follows that the shear in the panel $E F$ will at the same time be positive. The member $E f$ will then be in action and the stress in $E e$ will equal the shear on section s. But if this shear be less than the load at E, then the shear in panel $E F$ will be negative and $e F$ will be in action, making the stress in $E e$ equal to the load at E. Whence the rule, that the maximum total stress in $E e$ is equal to the maximum shear on section s, or to the maximum joint load at E, whichever is the greater. The minimum stresses in the verticals are found as in the through-bridge. For member $C c$, and in Fig. 50, also member $B b$, they are found from minimum shears; for members $D d$ and $E e$ they are equal to the dead joint load applied at the top.

The dead load stresses in the verticals may be found conveniently by first assuming all dead load applied at the upper chord joints and afterwards correcting the resulting stresses by *subtracting* therefrom the amount carried at each lower joint.

To illustrate the stress calculation for the verticals a truss will be assumed of the form shown in Fig. 50, but of the same dimensions and loading as in Art. 86. All the dead load will first be assumed applied at the upper joints. The shears in the several panels are the same as given in Art. 86, and are as follows:

Panel.	$A B$	$B C$	$C D$	$D E$
Maximum shears.......	+ 81,200	+ 60,000	+ 40,800	+ 23,600
Minimum shears........	+ 25,200	+ 16,000	+ 4,800	− 8,400

The maximum stresses will then be as follows:

$$B b = 81{,}200 \text{ (25,200 dead + 56,000 live).}$$
$$C c = 60{,}000 \text{ (18,000 dead + 42,000 live).}$$
$$D d = 40{,}800 \text{ (10,800 dead + 30,000 live).}$$

For member $E e$ the maximum shear in panel $D E$ (= shear on section s) $= 23{,}600$ lbs.; the total load at $E = P + W = 16{,}000 + 7{,}200 = 23{,}200$ lbs. The maximum stress in $E e$ is therefore 23,600 lbs.

The minimum stresses are as follows:

$B\,b$ = 25,200 (all dead load).
$C\,c$ = 16,000 (18,000 dead − 2,000 live).

For members $D\,d$ and $E\,e$ the minimum stress $= W$ = 7,200 lbs.

All of these stresses, both maximum and minimum, should now be decreased by $\frac{1}{3}W$, or 2,400 lbs., to allow for the portion of the dead load applied at the lower joints.

88. The Howe Truss.—(Fig. 51).—In this truss the chords and diagonals are of wood and the verticals of iron or steel. This type of truss has been extensively used in the past and is still employed to some

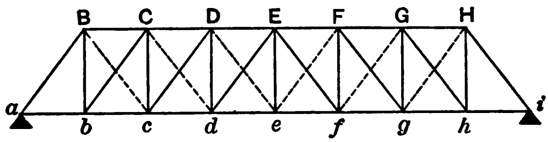


FIG. 51.

extent where timber is cheap and transportation difficult. It is so constructed that the diagonals can carry compression only, and the verticals tension. Main diagonals are shown by full lines, and counters by dotted lines. The figure shows counters in every panel, a form of construction frequently used. The counters near the ends of the bridge receive no definite stress, but they enable the details at the joints to be more satisfactorily designed.

The analysis of the Howe truss is exactly similar to that of the Pratt truss. In fact the stresses in the truss of Fig. 51 are numerically the same throughout as those in the deck Pratt truss of Fig. 50, the upper chord of Fig. 50 corresponding to the lower chord of Fig. 51, etc.

89. Determination of Chord Stresses by the Method of Chord Increments.—In horizontal chord trusses it is often convenient to determine the chord stresses by considering the horizontal components acting at each joint, beginning at the end of the structure. This method requires the shears and the horizontal components of the web stresses to be first determined. It will be explained by application to the example of Art. 86, Fig. 44. The dead-load chord stresses will be found.

The dead-load shears are given on p. 149. The horizontal com-

ponents of the diagonal stresses will be equal to the shears multiplied by tan θ, or by $16/20 = 0.8$. We therefore have

Member.	Shear.	Hor. Comp.
a B	25,200	20,160
B c	18,000	14,400
C d	10,800	8,640
D e	3,600	2,880

Then at joint *B*, Fig. 52, Σ hor. comp. = 0 gives the relation: S_4 = hor. comp. S_1 + hor. comp. S_3. Hence stress in $BC = 20{,}160 + 14{,}400 = 34{,}560$ lbs. Then at joint *C*, stress in $C\ D$ = stress in BC + hor. comp. $Cd = 34{,}560 + 8{,}640 = 43{,}200$ lbs.; and finally at joint *D*, stress in $DE = 43{,}200 + 2{,}880 = 46{,}080$ lbs. These check with the results found by moments. The stresses in the lower chords are found likewise. They are, in fact, given in the results already found.

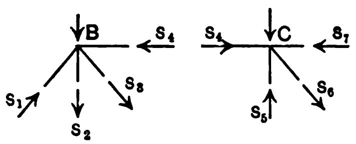


FIG. 52.

If this method is applied to live-load stresses it is to be noted that the shears or the web stresses needed are those due to a full load, which is the loading giving maximum chord stresses. They are not the maximum shears or maximum web stresses. The method is not so convenient in this case.

90. Use of Coefficients in Calculating Stresses.—In the analysis of numerous trusses of a similar type for uniform loads, it is a saving of time to calculate a set of coefficients for the various members, from which may be determined the stresses in any truss of a given number of panels by multiplying these coefficients by certain constants. These coefficients are the stresses or shears found by assuming the height, the panel length and the joint load each equal to unity. For web stresses the coefficients for dead- and maximum live-load will be different. The coefficients for a 7-panel Pratt truss are shown in Fig. 53. For web members the upper figure

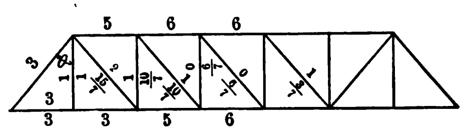


FIG. 53.

is the dead-load and the lower figure the live-load coefficient. Then for any similar truss of 7 panels the various stresses will be found by multiplying these coefficients by the following constants: For verticals, by the joint load W; for diagonals, by $W \sec \theta$; and for chord members, by $W \tan \theta$; in which W represents either dead- or live-joint load.

SECTION IV.—TRUSSES WITH INCLINED CHORDS AND SINGLE WEB SYSTEMS

91. Components of Stresses in Inclined Members.—In analyzing trusses with inclined chords it is frequently necessary to deduce the stress in a member, or its vertical or horizontal component, from another known component or from the stress itself. The simplest way to do this in practice is to make direct use of the length and the horizontal or vertical projection of the member in question, rather than to calculate angles and use trigonometrical tables. The desired relation of the components of stress is then obtained from the principle that the components of the stress are proportional to the corresponding projections of the member. Thus in Fig. 54, let $O\,A$ be any member of length l, and let l_x and l_y be its projections parallel to the axes $O\,X$ and $O\,Y$. Let S represent the stress in $O\,A$ and S_x and S_y the components of S resolved in directions parallel to $O\,X$ and $O\,Y$. Then by similar triangles we have

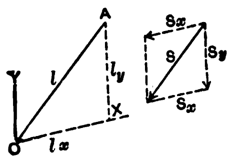


FIG. 54.

$$\left.\begin{aligned} S &= \frac{S_y}{l_y} \,.\, l = \frac{S_x}{l_x} \,.\, l, \\ S_x &= \frac{S_y}{l_y} \,.\, l_x = \frac{S}{l} \,.\, l_x, \text{ and} \\ S_y &= \frac{S_x}{l_x} \,.\, l_y = \frac{S}{l} \,.\, l_y. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From these equations the easily remembered rule may be stated: *To find any desired component divide the given component by the corresponding projection and multiply by the projection corresponding to the desired component.* This method will be generally used in the following work.

92. Chord Stresses.—Let $A\,B$, Fig. 55, be any truss with inclined chords. The simplest method of calculating the stress in any chord member, as $D\,E$, is by the method of moments, as in the horizontal

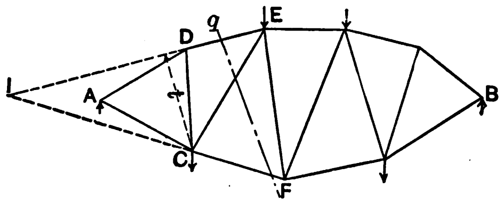


FIG. 55.

chord truss. The moment centre for $D\,E$ is at C. If its lever-arm is t, we may write, as in Art. 75,

$$S = \frac{M}{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $M = \Sigma$ moments about C of the external forces on the left of the section.

Instead of calculating the value of t it is often more convenient to get, first, the horizontal component of the chord stress and then reduce by the method explained in the preceding article. The horizontal component is found as follows: Fig. 56 shows the structure to the left

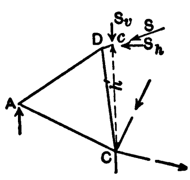


FIG. 56.

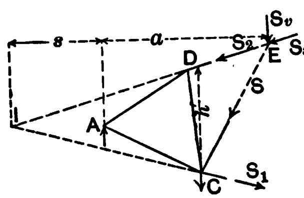


FIG. 57.

of the section. S is the unknown stress. Resolve this into the components S_v and S_h and apply these at a point c, vertically above C. Equating moments about C as before, S_v will not appear and we shall have

$$S_h = \frac{M}{h'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

That is, the horizontal component of the chord stress is equal to the bending moment divided by the vertical distance from the moment centre to the member. Then, from Art. 91,

$$S = \frac{S_h}{l_h} \cdot l = \frac{M}{h'} \cdot \frac{l}{l_h}. \quad \ldots \ldots \quad (4)$$

The chord stresses are a maximum when the bridge is fully loaded.

93. Web Stresses.—The stress in a web member of a truss, such as shown in Fig. 55, is obtained most easily by the method of moments. For member $C\,E$, for example, the moment centre is at the intersection of $D\,E$ and $C\,F$. The lever-arm of $C\,E$ may be measured from a large scale drawing of the truss.

So long as the chord members cut by the section intersect beyond the support, the maximum stress of either kind in the web member will be caused by a position of loads similar to that necessary in the horizontal chord bridge. If, however, the chord members intersect at a point between the supports, as in Fig. 58, the maximum stress will be caused by loading the truss fully. This form would rarely be used for a simple truss as it is not an economical type, being of minimum depth at point of maximum moment. The two-hinged arch, however, is often built in this form.

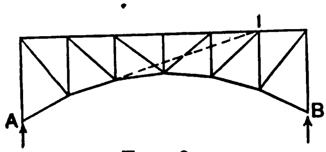


FIG. 58.

In a truss with one horizontal chord and with vertical web members (Fig. 59), the method of moments may conveniently be modified as follows: Suppose the stress in $E\,D$ be required. In Fig. (*a*) replace the stress S by S_h and S_v applied at point D. Then if M_i represents the moment of the external forces about I we have

$$S_v = \frac{M_i}{s + m\,d} \quad \ldots \ldots \ldots \quad (5)$$

and from Art. 91,

$$S = S_v \frac{l}{l_v} = \frac{M_i}{s + m\,d} \cdot \frac{l}{l_v}, \quad \ldots \ldots \quad (6)$$

in which l = length of $E\,D$, and l_v = vertical projection. All quantities are easily calculated, and if all lever-arms are expressed in panel lengths the work is still further simplified.

For the stress in the vertical $F\,D$ we have directly, from $\Sigma\,M = 0$ of the forces acting to the left of the section cutting $F\,D$,

$$S = \frac{M_i}{s + m\,d}. \qquad (7)$$

This method of moments can be applied also where all web members are inclined, but with less advantage. Thus in Fig. 57 the stress in $C\,E$ may be resolved into the vertical component S_v, and the component S_x parallel to the upper chord, and applied at point E. Then $S_v = M_i/(s + a)$, and $S = S_v\,l/h'$, where l = length of web member $C\,E$.

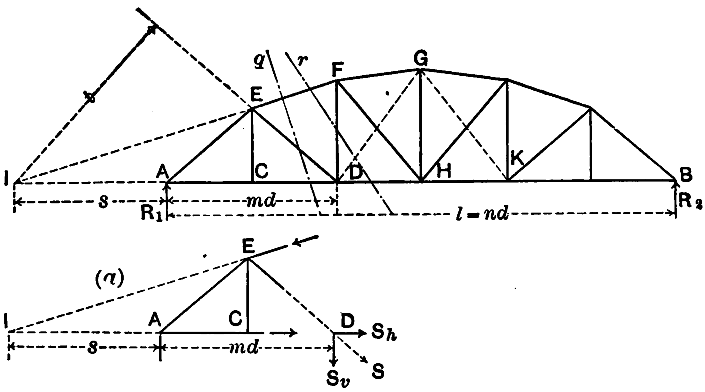


FIG. 59.

The method of shears is also readily applied to such a truss as shown in Fig. 59. We have here, as in the horizontal chord truss, the principle that the shear on any section, q, must be resisted by the vertical components of the stresses in the members cut. In this case, however, the upper chord member has a vertical component, and the vertical component in the web member will therefore equal the shear, minus or plus the vertical component in the chord member according as this vertical component acts in the opposite or the same direction as the shear.

The minimum stresses in a truss with counters, such as shown in Fig. 59, are found in the same general way as in the Pratt truss. The only point requiring notice is the calculation of the stress in the verticals which are adjacent to panels containing counters, such as member $F\,D$.

This receives its minimum stress when ED and DG are acting, in which case the stress is found by considering the forces acting at joint F. The chord members EF and FG have a resultant upward action which causes tension in FD, while the load at F causes compression. The resultant may be a small compressive stress or may be a tensile stress. The least compression or the greatest tension is what is wanted, and it will occur when the truss is as fully loaded as possible from the left without bringing FH into action. The required loading must be determined by trial by loading from the left up to D, then to H, then to K, and calculating the total stress in DG. The load is thus advanced until DG just becomes zero; FH will then also be zero. The middle vertical GH has its maximum tension under full load. The calculation of minimum stresses is illustrated in the example of Art. 96.

94. *Exact Method of Calculating Maximum Web Stresses for Moving Loads.*—In the foregoing discussion relative to maximum web stresses the assumption has been made of full joint loads on one side of the section and no loads on the other side, as in the conventional method of Art. 82. The exact analysis for maximum web stresses for a moving uniform load will now be given. As in Art. 82 let x (Fig. 60) = the

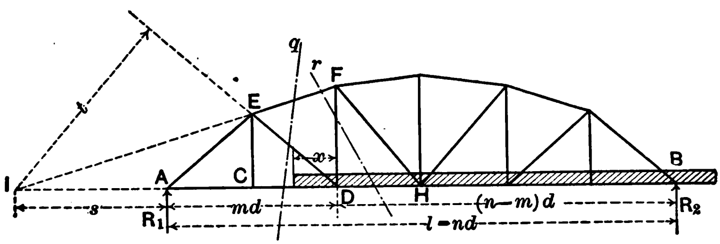


FIG. 60.

distance from the joint on the right of the section to the head of the load; m = the number of panels to the left of this joint, and n = whole number of panels. Let s = distance IA. Assume that the load has reached joint D in its movement towards the left and then consider the effect of a further movement into the panel DC. The stress in ED will be increased by adding increments of load to the left of D until we reach a distance x from D, at which point the addition of an increment will produce no additional stress. The stress in ED, due to an increment of load dP, placed a distance x from D, is found by taking mo-

ments about I. The joint load at C is $d\,P\,x/d$ and the reaction $=$ $R_1 = d\,P \,.\, \dfrac{x + (n - m)\,d}{l}$, whence the stress in $E\,D$ due to $d\,P$ is

$$d\,S = \frac{d\,P \,.\, \dfrac{x + (n - m)\,d}{l} \times s - d\,P\dfrac{x}{d}\,[s + (m - 1)\,d]}{t}$$

Placing this equal to zero, we derive

$$x = \frac{n - m}{n - 1 + n\,(m - 1)\,\dfrac{d}{s}}\,d. \quad . \quad . \quad . \quad . \quad (8)$$

Comparing this with the value of x given in Art. 82, eq. (8), we notice that here the denominator contains the additional term $n\,(m - 1)\,d/s$. This term becomes zero when $s = \infty$, or for parallel chords, as should be the case.

The stress in $E\,D$ for this loading is found by taking moments about I. If R_1 is the left abutment reaction, P' the joint load at C, and p the load per foot, the stress in $E\,D$ =

$$S = \frac{R_1 \times s - P' \times [s + (m - 1)\,d]}{t}.$$

Now

$$R_1 = \frac{p\,[\,(n - m)\,d + x]^2}{2\,l} \text{ and } P' = \frac{p\,x^2}{2\,d}.$$

Substituting the value of x from (4) and reducing, we have, for the maximum stress,

$$S = \frac{p\,d\,(n - m)^2}{2\,n}\left(1 + \frac{1}{n - 1 + n\,(m - 1)\,\dfrac{d}{s}}\right)\frac{s}{t}. \quad . \quad . \quad (9)$$

For the maximum stress in a vertical, $F\,D$, the load terminates in the panel $D\,H$. The distance x is now measured from joint H and m = number of panels to the left of this joint. The centre of moments is at I, and the lever-arm (value of t), is $(m - 1)\,d$.

95. Graphical Methods of Analysis.—The graphical method of analysis is well adapted to trusses with inclined chords. Chord stresses

may be determined either by the method of moments, using the equilibrium polygon as explained in Art. 73, or by the method of stress diagrams as used in roof truss analysis. A single stress diagram will serve to determine all dead-load chord and web stresses and another single diagram will give the live-load chord stresses. In constructing the dead-load stress diagram all counters should be omitted at first. The resulting diagram will give the chord stresses and the stresses in the main web members. A supplementary diagram should then be drawn with the counters in action and the main diagonals omitted. This will give the dead-load counter-stresses which will be needed in combining with the live-load stresses in these members. For live-load web stresses the method of stress diagrams is still very convenient if the conventional method of loading is adopted. For the more exact treatment the methods explained in the next chapter are better adapted.

In determining live-load web stresses a separate diagram must be drawn for each position of the load, a different position being required for each pair of diagonals meeting at the unloaded chord. This diagram need be drawn only as far as the members whose stresses are desired. The abbreviated diagram of Chapter II, p. 75, will be found to apply well here. The reactions for the several positions of the loads should first be computed, then laid off on the same vertical and all the diagrams drawn. Or the live-load web stresses may be found by a single diagram as follows: Assume a left abutment reaction of some convenient amount, as 100,000 lbs., and, with *no loads* on the truss, begin at the left and draw a stress diagram up to the last diagonal member on the right. The main diagonals are to be considered as acting on the left of the centre, and the counters on the right. Scale off and tabulate the stresses thus found. Compute the actual reactions for the various positions of the loads required. Then to get the maximum stress in any web member multiply the stress found from the diagram by the true reaction corresponding to the position of loads for a maximum stress in this member, and divide the result by 100,000. With a slide-rule this method is very rapid and easy. It is based on the fact that the diagram thus drawn and the diagrams drawn for each position of the loads are similar figures. This method is fully illustrated in Art. 100.

96. Analysis of a Curved-Chord Pratt Truss.—

EXAMPLE.—Let it be required to analyze the "Curved-Chord Pratt" truss shown in Fig. 61 for a uniform fixed load of 400 lbs. per foot per truss and a uniform moving load of 900 lbs. per foot per truss.

$$\text{Dead panel load} = W = 7{,}000 \text{ lbs.}$$
$$\text{Live panel load} = P = 15{,}750 \text{ lbs.}$$
$$\text{Dead-load reaction} = \frac{7W}{2} = 24{,}500 \text{ lbs.}$$

All the dead load will at first be assumed as applied at the lower joints. Panel length $= d = 17.5$ ft.

Chord Stresses.—Applying the method of Art. 92, eq. (3), we may write at once:

$a\,b$ and $b\,c = (24{,}500 \times 1)\ 17.5/18 = 23{,}800.$

$c\,d$ and hor. comp. $B\,C = (24{,}500 \times 2 - 7{,}000 \times 1)\ 17.5/22 = 33{,}400.$

$d\,e$ and hor. comp. $C\,D = [24{,}500 \times 3 - 7{,}000 \times (1 + 2)]17.5/24 = 38{,}000.$

Hor. comp. $D\,E = [24{,}500 \times 4 - 7{,}000\,(1 + 2 + 3)]\ 17.5/25 = 39{,}200.$

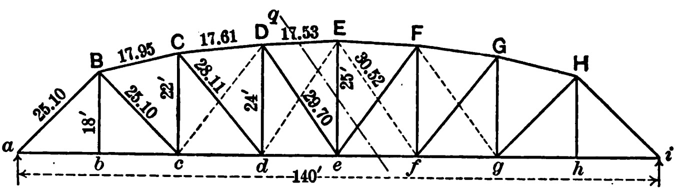


FIG. 61.

The stresses in the upper chord members are found from their horizontal components as follows: $B\,C = 33{,}400 \times 17.95/17.5 = 34{,}300$; $C\,D = 38{,}000 \times 17.61/17.5 = 38{,}200$; $D\,E = 39{,}200$; $\times\ 17.53/17.5 = 39{,}300$.

The live-load stresses are equal to the above dead-load stresses multiplied by $\frac{900}{400}$ or 2.25.

Web Stresses.—(*a*) *Dead Load.*—These will be found by the method of shears, getting the vertical components of the web stresses by subtracting from the shears the vertical components of the chord stresses. The shears are:

Panel $a\,b = +\ 24{,}500$,	Panel $d\,e = +\ 3{,}500$,
Panel $b\,c = +\ 17{,}500$,	Panel $e\,f = -\ 3{,}500$,
Panel $c\,d = +\ 10{,}500$,	etc.

The vertical components of the chord stresses will be found from their horizontal components already calculated. They are: Vert. comp. $B\,C = 33{,}400 \times 4/17.5 = 7{,}600$; Vert. comp. $C\,D = 38{,}000 \times 2/17.5 = 4{,}300$; Vert. comp. $D\,E = 39{,}200 \times 1/17.5 = 2{,}200$.

The vertical components of the stresses in the main diagonals are therefore:

Vert. comp. $a\,B = 24{,}500.$

Vert. comp. $B\,c = 17{,}500 - 7{,}600 = 9{,}900.$

Vert. comp. $C\,d = 10{,}500 - 4{,}300 = 6{,}200.$

Vert. comp. $D\,e = 3{,}500 - 2{,}200 = 1{,}300.$

The stresses will be:

$$a\,B = 24{,}500 \times 25.10/18 = 34{,}200.$$
$$B\,c = 9{,}900 \times 25.10/18 = 13{,}800.$$
$$C\,d = 6{,}200 \times 28.11/22 = 7{,}900.$$
$$D\,e = 1{,}300 \times 29.70/24 = 1{,}600.$$

All are tensile except $a\,B$.

When the counter $d\,E$ is in action the dead-load stress in it is ***compression***, and is calculated just as if $D\,e$ did not exist. Its vertical component will equal the shear in panel $d\,e$ minus the vertical component of the simultaneous stress in $D\,E$. Now when $d\,E$ is acting the moment centre for $D\,E$ is at d and the horizonal component of its stress will equal Moment at $d \div 24$, which is the same as the horizontal component in $C\,D$ already computed, $=$ 38,000 lbs. The vertical component in $D\,E = 38{,}000 \times 1/1.75 =$ 2,150 lbs. Finally vert. comp. $d\,E = 3{,}500 - 2{,}150 = 1{,}350$ lbs., and the stress $=$ $1{,}350 \times 30.52/25 = 1{,}600$ lbs. compression.

In the same manner we have, for the dead-load compression in $c\,D$: Hor. comp. $C\,D =$ hor. comp. $B\,C = 33{,}400$; vert. comp. $C\,D = 33{,}400 \times 2/17.5 = 3{,}800$; vert. comp. $c\,D = 10{,}500 - 3{,}800 = 6{,}700$; stress in $c\,D = 6{,}700 \times 29.70/24 = 8{,}300$ lbs. compression. In case the live-load tension in $c\,D$ does not exceed the dead-load compression, then this member is not needed.

The dead-load stresses in the verticals (main diagonals acting) are:

$$B\,b = 7000 \text{ lbs. tension.}$$
$$C\,c = 10{,}500 - 7{,}600 = 2{,}900 \text{ lbs. compression.}$$
$$D\,d = 3{,}500 - 4{,}300 = 800 \text{ lbs. tension.}$$

The member $E\,e$ will receive its maximum compression when $E\,f$ (or $d\,E$) is under maximum stress, and the value of its stress will then be equal to the shear on section q minus or plus the vertical component in $D\,E$, Fig. 62. The dead-load shear $= -3{,}500$

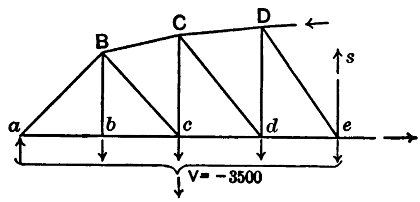


FIG. 62.

lbs., and the vert. comp. in $D\,E = 2{,}200$ lbs. compression. For equilibrium the tension $E\,e$ must equal $3{,}500 + 2{,}200 = 5{,}700$ lbs.

We will now correct the stresses in the verticals for $\frac{1}{3}\,W$ which will be assumed applied at the upper joints, giving finally the stresses

$$B\,b = -\ 7{,}000 + 2{,}300 = 4{,}700 \text{ lbs. tension.}$$
$$C\,c = 2{,}900 + 2{,}300 = 5{,}200 \text{ lbs. compression.}$$
$$D\,d = -\ 800 + 2{,}300 = 1{,}500 \text{ lbs. compression.}$$
$$E\,e = -\ 5{,}700 + 2{,}300 = 3{,}400 \text{ lbs. tension.}$$

(*b*) *Live Load.*—The maximum stress in $a\,B =$ dead-load stress $\times$ 900/400 $=$ 76,900 lbs. Stress in $B\,b = P = 15{,}750$ lbs. For the remaining members the method of moments

explained in Art. 93 will be used. The distances from the left support to the points of intersection of the various upper chord segments with the lower chord are as follows:

for $B\,C$, 3.5 panel lengths,
for $C\,D$ 9 " "
for $D\,E$, 21 " "

For the maximum stress in $B\,c$, joints c to h are loaded, and the left reaction = $P/8\,(1 + 2 + 3 + 4 + 5 + 6) = 41{,}300$ lbs. Then in eq. (5), Art. 93, $M = R_1\,s$ and we have

$$\text{Vert. comp. } B\,c = R_1 \times \frac{s}{s + m\,d} = 41{,}300 \times \frac{3.5}{5.5} = 26{,}300.$$

In like manner, for the other diagonals

$$\text{Vert. comp. } C\,d = \frac{P}{8}(1 + 2 + 3 + 4 + 5)\frac{9}{12} = 22{,}100.$$

$$\text{Vert. comp. } D\,e = \frac{P}{8}(1 + 2 + 3 + 4)\frac{21}{25} = 16{,}500.$$

For the counter $E\,f$ the load extends to f, and the centre of moments is 21 panels to the right of point i or 29 panels to the right of R_1. We have then

$$\text{Vert. comp. } E\,f = \frac{P}{8}(1 + 2 + 3)\frac{29}{24} = 14\,300.$$

Likewise

$$\text{Vert. comp. } F\,g = \frac{P}{8}(1 + 2)\frac{17}{11} = 9{,}100.$$

The stresses in the diagonals are

$B\,c = 26{,}300 \times 25.10/18 = 36{,}700$; $C\,d = 22{,}100 \times 28.11/22 = 28{,}200$;
$D\,e = 16{,}500 \times 29.70/24 = 20{,}400$; $E\,f = 14{,}300 \times 30.52/25 = 17{,}500$;
$F\,g = 9{,}100 \times 29.70/24 = 11{,}300$.

The stresses in the verticals are calculated in a similar manner as follows:

$$C\,c = \frac{P}{8}(1 + 2 + 3 + 4 + 5)\frac{3.5}{5.5} = 18{,}800.$$

$$D\,d = \frac{P}{8}(1 + 2 + 3 + 4)\frac{9}{12} = 14{,}800.$$

$$E\,e = \frac{P}{8}(1 + 2 + 3)\frac{21}{25} = 9{,}900.$$

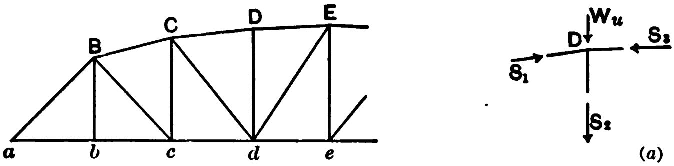
All these stresses are compressive.

The maximum compression in the verticals and the maximum tension in all diagonals may now be made up as follows, using signs to indicate the kind of stress:

Member.	Dead Load.	Live Load.	Total or Maximum.
$a\,B$	− 34,200	− 76,900	− 111,100
$B\,b$	+ 4,700	+ 15,700	+ 20,400
$B\,c$	+ 13,800	+ 36,700	+ 50,500
$C\,c$	− 5,200	− 18,800	− 24,000
$C\,d$	+ 7,900	+ 28,200	+ 36,100
$D\,d$	− 1,500	− 14,800	− 16,300
$D\,e$	+ 1,600	+ 20,400	+ 22,000
$E\,e$	+ 3,400	− 9,900	− 6,500
$E\,f$	− 1,600	+ 17,500	+ 15,900
$F\,g$	− 8,300	+ 11,300	+ 3,000

There still remains to be determined the maximum tension in the verticals $C c$, $D d$, and $E e$. The minimum stress in all diagonals except $B c$ is zero. That in $B c$ occurs with joint b loaded. It can readily be found if needed.

The maximum tension in $E e$ occurs for a fully loaded bridge, as in that case members $D e$ and $e F$ will be in action. The dead-load tensile stress = vert. comp. $D E$ + vert.



(a)

FIG. 63.

comp. $E F - \frac{1}{2} W$ (applied at E) = 2,200 + 2,200 − 2,300 = 2,100 lbs. The live-load vert. comp. in $D E$ = (dead load) × 900/400 = 2,200 × 900/400 = 4,900. The live-load tension in $E e$ is therefore 4,900 + 4,900 = 9,800 lbs. Total tension = 2,100 + 9,800 = 11,900 lbs.

To get the maximum tension in $D d$, members $C d$ and $d E$ are assumed to be in action, (Fig. 63). The dead-load tensile stress in $D d$ is then equal to vert. comp. $C D$ − vert. comp. $D E$ − 2,300 lbs. (load at D). Now with $C d$ and $d E$ acting, the horizontal component in $D E$ is equal to the horizontal component in $C D$ of 38,000 lbs. already found, whence vert. comp. $D E$ = 38,000 × 1/17.5 = 2,200 lbs. We then have tensile stress in $D d$ = 4,300 − 2,200 − 2,300 = 100 lbs., or practically zero. The live-load tension in $D d$ will be a maximum when the bridge is as fully loaded as possible and yet have $d E$ in action instead of $D e$. This may be determined by trial. The member $d E$ is in action when joints b, c and d are loaded, as it then receives its maximum stress. A load now placed at joint e, or at any point to the right, will increase the chord stresses $C D$ and $D E$, and will therefore increase the tension in $D d$ so long as $d E$ remains in action. As soon as $D e$ comes into action any further load to the right of panel $d e$ will tend to cause compression in $D d$ or reduce its tension. The effect of loads at e and f may be separately considered. A load of P = 15,750 at e causes a compression in $d E$ (using the same method of calculation as before) equal to

$$\frac{15{,}750}{8} \times 4 \times \frac{21}{24} \times \frac{30.52}{25} = 8{,}400 \text{ lbs.}$$

A load at f causes a compression equal to three-fourths of this amount = 6,300 lbs. The total stress (dead and live), in $d E$, with bridge loaded to d, has been found to be 15,900 lbs. tension. When loaded to e it will therefore be equal to 15,900 − 8,400 = 7,500 lbs. tension, and when loaded to f it will be 7,500 − 6,300 = 1,200 lbs. tension. Therefore this member will still be in action for joint f loaded, but it is evident that to load joint g in addition would cause the resultant stress to be compression, that is, it would throw $D e$ into action. For a maximum tension in $D d$, therefore, the truss is to be loaded from b to f. Under this load the value of R_1 = 49,200 lbs., and the horizontal components in $C D$ and $D E$ = $[49{,}200 \times 3 - 15{,}750 (2 + 1)] \times \frac{17.5}{24}$. The vert. comp. in $C D$ = hor. comp. × 2/17.5 = 8,400 lbs., and vert. comp. in $D E$ = hor. comp. × 1/17.5 = 4,200 lbs., whence the live-load tension in $D d$ = 8,400 − 4,200 = 4,200 lbs. Total tension = 4,200 + 100 = 4,300 lbs.

By the same method it can be shown that for the maximum tension in $C c$ only joints b and c should be loaded. The dead-load tensile stress = vert. comp. $B C$ − vert. comp. $C D - \frac{1}{3} W$ = 1,500 lbs. The live-load tension = 3,200 lbs. Total = 4,700 lbs.

97. Other Forms of Curved-Chord Trusses.—The curved-chord Pratt truss as illustrated above is the most common form of curved-chord truss in use at the present time. In the deck-truss the upper chord is made horizontal and the lower chord curved. Formerly, extensive use was made of the Parabolic Bowstring truss, in which the joints of the upper chord lie on the arc of a parabola. Since the moment curve for a uniform load is a parabola, it follows that the horizontal components of the upper chord stresses and the stresses throughout the lower chord are uniform. The diagonals therefore receive no stress under uniform load, but for moving loads counters are required in every panel. The uniformity of chord stress was a point in favor of this truss, but the difficulty of making the wind bracing effective at the ends of the bridge and the extensive use of adjustable counters were objections.

98. *The Double-bowstring or Lenticular Truss* (Fig. 64) has both

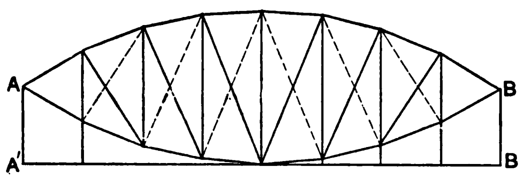


FIG. 64.

chords in the form of a parabola. The floor may be supported along the centre line $A B$, or may be hung below, along the line $A' B'$. In the latter case a horizontal wind-truss in the plane $A' B'$ prevents the swaying of the main truss longitudinally. With verticals and diagonals as in the figure, the horizontal component of the chord stress is constant, for the sums of the ordinates to two parabolas give the ordinates to a third parabola.

99. *The Schwedler Truss* is a form of curved-chord truss in which the slopes of the upper chord members in certain of the panels next to the end is made as great as possible without requiring the use of a counter in the panel. That is, the stress in the main diagonal does not become quite as small as zero. The necessary slope depends on the

relation of live to dead load and can be found by placing equal to zero the total calculated diagonal stress when the truss is so loaded as to cause the minimum live-load stress. Obviously these conditions cannot be met near the centre of the bridge as counters will be required here even with horizontal chords. While it is advantageous to have as large slope for the chords as possible without the use of many counters, other practical considerations will usually control the design, so that the precise proportions of the Schwedler truss are seldom used.

100. *The Pegram Truss* (Fig. 65) illustrates certain principles of economical design. It is so proportioned that the compression members are in general shorter than the tension members; the upper chord members are of equal length and have nearly equal stresses to carry; and the compressive diagonals are shorter near the ends where the stresses are a maximum. The chief objection to this type of truss is the difficulty and expense of making satisfactory riveted connections between the floor system and the truss, in accordance with good modern practice. It has practically gone out of use, but as its analysis serves very well to illustrate methods especially applicable to trusses with inclined chords and inclined web members, an example of its analysis will be given.

EXAMPLE.—Take a 200-ft. through-span, Fig. 65, with seven panels, each equal to 28.57 ft. The coordinates of the upper panel points are given in the figure. These points lie in a circular arc with a chord of 160 ft. and versed sine of 15 ft. Each top chord member between pins is 23.55 ft. long except the centre one, which is 1/20 less or 22.37 ft. long. This is made shorter to enable the chord sections between splices to be of uniform length, the splices being placed towards the end of the truss from the pin-points.

The dead load will be assumed at 875 lbs. per foot per truss; the live load at 1,800 lbs. per foot per truss. The dead panel load = 875 × 28.57 = 25,000 lbs. It will be considered as all applied at the lower joints. Live panel load = 1,800 × 28.57 = 51,430 lbs. The stresses will be found by the graphical method.

Dead-load Stresses.—The abutment reaction, R_1, = 3 × 25,000 = 75,000 lbs. Laying off $B\ A$, Fig. 67, equal to this, and $A\ P$, $P\ Q$, and $Q\ R$ each equal to 25,000 lbs., we draw the truss diagram for one-half the truss as usual. The counter 5–8 is considered at first as not in action and the diagram drawn as shown by the full lines. This gives the dead-load stresses in all chord members and the dead-load stresses in the web members which obtain when the truss is loaded for maximum positive shear. These stresses are scaled off and written in the diagram of Fig. 66 and marked "D." To get the dead-load stress in the counter 5–8, when the truss is loaded for maximum negative shear, the diagram must be drawn under the assumption that 5–8 is acting instead of 6–7. This is shown by the dotted lines in Fig. 67, and the stress in 5–8 is given by $G'\ H'$. The same

diagram gives the dead-load tension ($G' F$), in member 5–6 when the counter is in action. As a check on the diagram the stress in 8–10 is calculated by moments to be

$$\frac{875 \times (28.57)^2 \times 3 \times 4}{2 \times 38.72} = 110{,}700 \text{ lbs.}$$

Live-load Stresses.—The stresses in the chords, and in 1–2 and 2–3 are a maximum for full load, and may therefore be obtained by multiplying the corresponding dead-load stresses by 1,800/875.

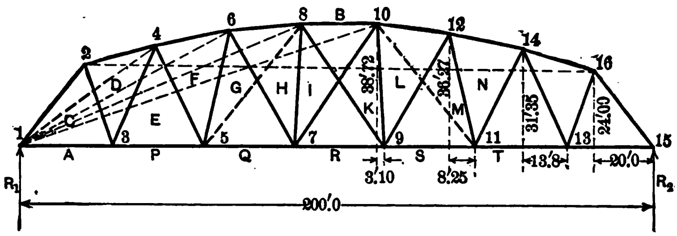


FIG. 65.

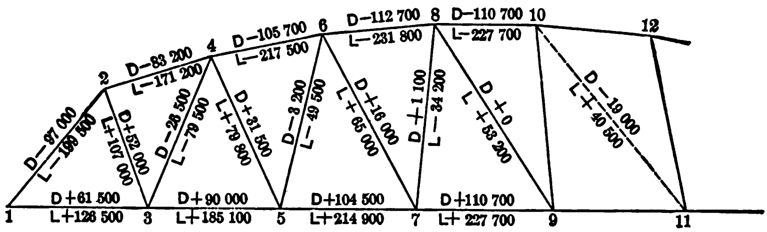


FIG. 66.

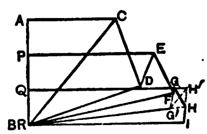


FIG. 67.

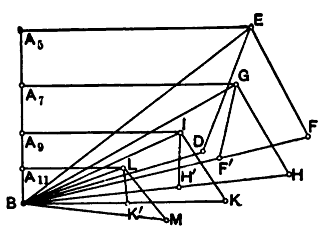


FIG. 68.

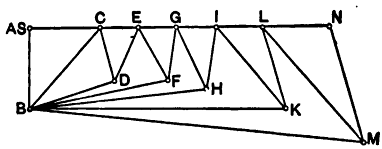


FIG. 69.

For a maximum in 3–4 and 4–5 all joints up to 5 should be loaded. The reaction R_1 is then equal to $pd/7$ (1 + 2 + 3 + 4 + 5) = 110,200 lbs. Laying this off as $B\,A_5$, Fig. 68, we proceed to draw the diagram as far as piece 4–5 by the method explained in

Art. 50. Substituting the triangle 1–4–5 for the original framework, we find the stress in 4–5, or $E F$, by drawing $A_5\ E$ parallel to 1–5 and $B E$ parallel to 4–1; then $E F$ parallel to 4–5 and $B F$ parallel to 4–6; whence $E F$ is the required stress in 4–5. To find the stress in 3–4, draw the diagram for joint 4 of the original truss. This diagram is $B\ D\ E\ F\ B$, the portion $E\ F\ B$ being already drawn; $D\ E$ is the stress in 3–4.

For a maximum in 5–6 and 6–7 all joints but 3 and 5 should be loaded. The reaction $R_1 = pd/7\ (1 + 2 + 3 + 4) = 73{,}470$ lbs. Laying off $B\ A_7$ equal to this, we proceed as for 3–4 and 4–5. Substituting the triangle 1–6–7 for the portion of the truss to the left of 7, the diagram $B\ A_7\ G$ and thence $G\ B\ H$ determines the stress in 6–7, or $G\ H$. The diagram for joint 6 is all drawn except the line $G\ F'$. This drawn gives the stress in the post 5–6. The stresses in the other web members are found in like manner. The last loaded panel in each case is indicated by the subscript to the letter A in the diagram. The stresses are given in Fig. 66, marked "L."

The live-load web stresses may also be found by diagram as explained in Art. 95; that is, by assuming a reaction of 100,000 lbs., drawing the corresponding diagram and finding the actual stresses from this diagram by proportion. Fig. 69 is such a diagram, with $B\ A$ = 100,000 lbs. The diagram is drawn on the assumption that members 8–9 and 10–11 are in action, as would be the case for positive shear.

The few computations may be tabulated thus:

LIVE-LOAD WEB STRESSES.

Member.	Stress from Diagram. R_1 = 100,000 lbs.	Actual Reaction.	Actual Stress.
(1)	(2)	(3)	(4)
3–4	72,100	110,200	79,500
4–5	72,500	110,200	79,900
5–6	67,000	73,500	49,200
6–7	88,700	73,500	65,200
7–8	77,500	44,100	32,400
8–9	121,000	44,100	53,200
10–11	185,000	22,000	40,700

Column (3) contains the actual reactions when the truss is loaded so as to produce the maximum stresses in the corresponding members of Column (1). Column (4) is obtained by multiplying the quantities in Column (2) by those in (3) and dividing by 100,000. The resulting stresses should be the same as those found from Fig. 68.

If analytical methods are preferred, the same general methods are to be used as given in Arts. 92 and 93, i. e., the chord stresses found by moments and the wei stresses by moments or shears. In panel 5–7 the compression in 5–8 is to be found for dead load by assuming 6–7 as not acting, for the same reason as given in the above analysis.

In the analysis here given the work has been carried only far enough to get maximum stresses. For miminum stresses in some of the web members some additional diagrams would be needed in accordance with the treatment of Art. 96.

Section V.—Trusses with Multiple Web Systems

101. The Whipple Truss, shown in Fig. 70, consists of two simple Pratt trusses combined. It is in fact often called a "double intersection" Pratt truss. The advantage of this form over the Pratt truss for long spans is in having short panels, and yet an economical inclination for the diagonals (about 45°). It has been used extensively in the past, but the form shown in Fig. 77 or Fig. 84 is now preferred.

The two systems of web members, distinguished by full and dotted lines, are commonly assumed to act independently. This is not, however, strictly true, for the chords connecting the systems, while allowing independent vertical deflection, do not allow independent horizontal displacements, so that the loads on one system affect to some extent the form of, and hence the distribution of stress in, the other system. The exact analysis can only be made by the theory of redundant members given in Chapter VII, and even then it depends upon the adjustment of the counters. The assumption of independent systems is, however, very nearly correct and enables the stresses to be statically determined. In any case the question affects the web stresses only, as the chord stresses are a maximum for a full load, and under that condition the usual assumptions are correct.

The truss of Fig. 70 will be analyzed for the following loading: dead load = 500 lbs. per foot per truss; live load = 800 lbs. per foot per

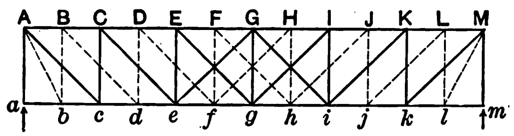


FIG. 70.

truss. The bridge will be assumed to be a through-bridge. Figs. 71 and 72 show the two systems separated. The joint load on each system = $500 \times 15 = 7{,}500$ lbs. dead load, and $800 \times 15 = 12{,}000$ lbs. live load.

The analysis can be made exactly as in the Pratt truss, treating each system of bracing, with the chords, as a separate truss. The web stresses so found will be correct. The resultant stress in any chord member is found by adding the stresses obtained from the two separate

trusses. Thus the total stress in member *D E*, for example, is equal to the stress in *C E* of Fig. 71, plus the stress in *D F* of Fig. 72.

The chord stresses may also be found directly, from Fig. 70, by noting that under a full uniform load none of the diagonals meeting the upper chord at joints *F*, *G*, and *H* are in action. The stress in member *F G* or *G H* can then be found at once by equating moments about *g*.

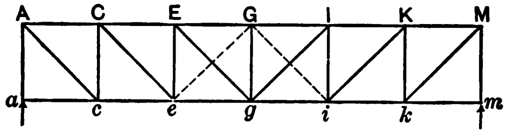


FIG. 71.

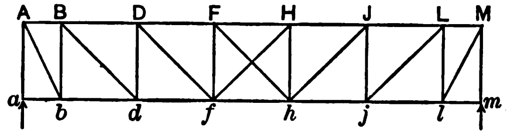


FIG. 72.

The stresses in the remaining upper chord members can then be found by the method of chord increments, Art. 89. Then for the lower chord stresses note that stress in *f g* = stress in *D E*, etc. Stress in *b c* = stress in *c d* minus hor. comp. in *A c*; stress in *a b* = stress in *b c* minus hor. comp. in *A b*, and should equal zero. This method of calculation will be used here.

Chord Stresses.—Dead-load stress in $F\,G = \dfrac{w\,d^2}{2\,h}\,(6 \times 6) =$ 67,500 lbs.

The hor. comp. in a diagonal = vert. comp. multiplied by $2\,d/h$, or, in the case of the diagonals *A b* and *l M*, by d/h. The vert. comp. = shear in the panel of the system to which the diagonal belongs. We have then for Fig. 71, since $2\,d = h$,

$$\text{Hor. comp. } E\,g = 1/2\,w\,d = 3{,}750;$$
$$\text{Hor. comp. } C\,e = 3/2\,w\,d = 11{,}250;$$
$$\text{Hor. comp. } A\,c = 5/2\,w\,d = 18{,}750.$$

For Fig. 72,

$$\text{Hor. comp. } D\,f = w\,d = 7{,}500;$$
$$\text{Hor. comp. } B\,d = 2\,w\,d = 15{,}000;$$
$$\text{Hor. comp. } A\,b = \tfrac{1}{2} \times 3\,w\,d = 11{,}250.$$

We have then, by subtraction, the chord stresses as follows:

$$
\begin{aligned}
EF &= 67{,}500;\\
DE = fg &= 67{,}500 - 3{,}750 = 63{,}750;\\
CD = ef &= 63{,}750 - 7{,}500 = 56{,}250;\\
BC = de &= 56{,}250 - 11{,}250 = 45{,}000;\\
AB = cd &= 45{,}000 - 15{,}000 = 30{,}000;\\
bc &= 30{,}000 - 18{,}750 = 11{,}250.
\end{aligned}
$$

The stress in $b\,c$ is found to be equal to the horizontal componen t o $A\,b$, which proves the correctness of the work.

The live-load chord stresses are obtained, as before, by proportion.

Web Stresses.—The dead-load vertical components, or shears in the separate systems, are given above.

For live-load shears, $p\,d/6 = 2{,}000$, and we have for Fig. 71:

$$
\begin{aligned}
\text{Shear in } a\,c &= (p\,d \times 5) \div 2 = 30{,}000;\\
\text{Shear in } c\,e &= 2{,}000\,(1 + 2 + 3 + 4) = 20{,}000;\\
\text{Shear in } e\,g &= 2{,}000\,(1 + 2 + 3) = 12{,}000;\\
\text{Shear in } g\,i &= 2{,}000\,(1 + 2) = 6{,}000.
\end{aligned}
$$

For Fig. 72, $p\,d/12 = 1{,}000$, and

$$
\begin{aligned}
\text{Shear in } a\,b &= (p\,d \times 6) \div 2 = 36{,}000;\\
\text{Shear in } b\,d &= 1{,}000\,(1 + 3 + 5 + 7 + 9) = 25{,}000;\\
\text{Shear in } d\!f &= 1{,}000\,(1 + 3 + 5 + 7) = 16{,}000;\\
\text{Shear in } f\!h &= 1{,}000\,(1 + 3 + 5) = 9{,}000;\\
\text{Shear in } h\,j &= 1{,}000\,(1 + 3) = 4{,}000.
\end{aligned}
$$

Adding to the above the dead-load shears, we have the stresses in the verticals, and the vertical components of the stresses in the diagonals. Vertical component of stress in the counter $G\,i$ = shear in $g\,i = 6{,}000 - 3{,}750 = 2{,}250$ lbs. This is also the maximum compression in $G\,g$. The vertical component of stress in $F\,h$ or $f\,H$ = 9,000 lbs. = stress in $F\!f$ and $H\,h$. There is no positive shear in $h\,j$, and hence no counter is needed. The stresses in $A\,a$ and $M\,m$ are equal to the sum of the shears in $a\,b$ and $a\,c$.

The stresses in the verticals can be corrected for the loads carried at the upper joints, as in the Pratt truss.

Only maximum web stresses have here been found. The student

will have no difficulty in extending the analysis to the determination of all the minimum stresses.

Fig. 73 shows a more common arrangement of the end posts than that in Fig. 70. This arrangement, however, causes a further ambiguity in stress calculation from not knowing to which truss system the loads at C and D belong. Assuming them equally divided between the two systems is nearly correct and enables the stresses to be readily

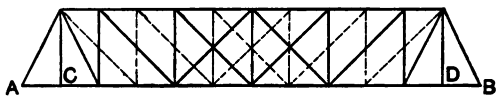


FIG. 73.

found. Each truss will then contain the vertical members at C and D, and the joint loads at these points will be one-half the full joint load. If an odd number of panels be used the ambiguity of calculation is again increased as the two systems into which the web members divide themselves under partial loads are unsymmetrical with respect to the centre.

The uncertainty of computations of stresses in double systems by the usual methods constitutes a somewhat serious defect in such systems, and is one cause that has led to the adoption of the forms referred to at the beginning of this article.

102. The Triple-intersection Truss has been built to some extent. It is similar to the Whipple truss, but has three instead of two sets of web members. The stresses are found in the same way as in the Whipple truss.

103. The Double-triangular Truss, shown in Fig. 74, has two systems of triangular bracing. This truss is used both as a short-span

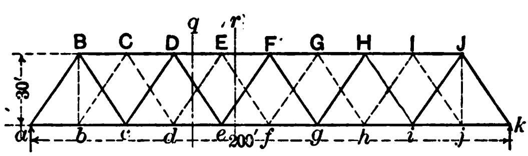


FIG. 74.

riveted structure and as a long-span pin-connected bridge. The Memphis bridge and the Kentucky and Indiana bridge are of this form, though modified as shown in Fig. 86. In analyzing this truss it is

divided into the two systems as shown in Figs. 75 and 76. Any joint load at B and J would be assumed equally divided between the two systems.

EXAMPLE.—(Fig. 74). Span length = 200 ft., height = 30 ft., w = 1,200 lbs. per ft. per truss, p = 2,000 lbs. per ft. per truss. W = 1,200 × 20 = 24,000 lbs. Assume each lower joint load = 16,000 lbs., and each upper joint load, including joints B and J, = 8,000 lbs. P = 40,000 lbs. The stresses in the eight members cut by sections q and r will be found.

Chord Stresses.—Dead-load reaction (Fig. 75) = (4 × 16,000 + 4 × 8,000) ÷ 2 = 48,000; for Fig. 76 it is (5 × 16,000 + 5 × 8,000) ÷ 2 = 60,000. Live-load reactions are respectively 80,000 and 100,000. Dead-load stress in $D\,F$ (Fig. 75) = (48,000 × 2 − 16,000 − 4,000 × $1\frac{1}{2}$ − 8,000 × $\frac{1}{2}$) 40/30 = 93,300. Likewise are found 80,000 in $c\,e$

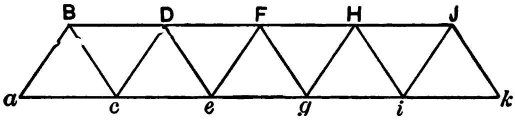


FIG. 75.

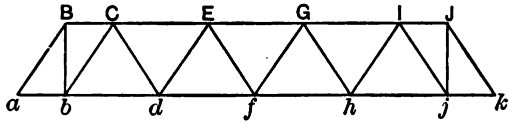


FIG. 76.

and 96,000 in $e\,g$. Stress in $C\,E$ (Fig. 76) = (60,000 × $1\frac{1}{2}$ − 16,000 − 4,000 − 8,000 × $\frac{1}{2}$) 40/30 = 88,000; stress in $E\,G$ = 104,000; stress in $d\,f$ = 98,700. The live-load stresses are found to be: $D\,F$ = 160,000, $c\,e$ = 133,300, $e\,g$ = 160,000, $C\,E$ = 146,700, $E\,G$ = 173,300 and $d\,f$ = 160,000. The required stresses are therefore as follows:

Dead-load stresses

$$
\begin{aligned}
D\,E &= 88{,}000 + 93{,}300 = 181{,}300,\\
E\,F &= 93{,}300 + 104{,}000 = 197{,}300,\\
d\,e &= 80{,}000 + 98{,}700 = 178{,}700,\\
e\,f &= 98{,}700 + 96{,}000 = 194{,}700.
\end{aligned}
$$

Live-load stresses

$$
\begin{aligned}
D\,E &= 160{,}000 + 146{,}700 = 306{,}700,\\
E\,F &= 160{,}000 + 173\ 300 = 333{,}300,\\
d\,e &= 133{,}300 + 160{,}000 = 293{,}300,\\
e\,f &= 160{,}000 + 160{,}000 = 320{,}000.
\end{aligned}
$$

Web Stresses.—The dead-load shears on the sections, or the vertical components of the diagonal stresses are:

$$
\text{Fig. 75}\begin{cases} e\,F = 4{,}000,\\ D\,e = 4{,}000 + 16{,}000 = 20{,}000.\end{cases}
$$

$$
\text{Fig. 76}\begin{cases} E\,f = 8{,}000,\\ d\,E = 8{,}000 + 8{,}000 = 16{,}000.\end{cases}
$$

The live-load positive shears are:

$$\text{Fig. 75}\begin{cases} eF = 1/5\,P\,(1+2) = 24{,}000. \\ De = 1/5\,P\,(1+2+3) = 48{,}000, \end{cases}$$

$$\text{Fig. 76}\quad dE = Ef = 1/10\,P\,(1+3+5) = 36{,}000.$$

The live-load negative shears are:

$$\text{Fig. 75}\begin{cases} De = 1/5\,P = 8{,}000, \\ eF = 1/5\,P\,(1+2) = 24{,}000, \end{cases}$$

$$\text{Fig. 76}\quad dE = Ef = 1/10\,P\,(1+3) = 16{,}000.$$

The resulting maximum and minimum shears and stresses are as follows:

Member.	Shears.					Stresses.	
	+ Dead.	+ Live.	− Live.	Max.	Min.	Max.	Min.
De	20,000	48,000	8,000	+ 68,000	+ 12,000	+ 81,700	+ 14,400
eF	4,000	24,000	24,000	+ 28,000	− 20,000	− 33,700	+ 24,000
dE	16,000	36,000	16,000	+ 52,000	0	− 62,500	0
Ef	8,000	36,000	16,000	+ 44,000	− 8,000	+ 52,900	− 9,600

104. The Lattice Truss, Fig. 77, has commonly four systems of bracing. It is seldom used for long spans and is always built as a riveted structure. It is best analyzed by treating the four systems as independent, although, as the web members are riveted together at their points of intersection, independent action is not possible. For small trusses it is customary to analyze the truss as a beam. The maximum moment and shear is found at several different sections; the

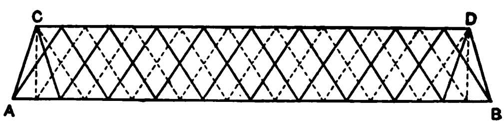


FIG. 77.

stresses in the chords or flanges are found by assuming them to take all the moment, and those in the web members by assuming the shear as equally divided among the members cut.

105. Double Intersection Trusses with Curved Chords.—In double-intersection trusses with curved upper chords each truss system is affected by loads on the other system, owing to the uplifting force at each angle of the upper chord. The systems are, as a result, not so independent as in the horizontal chord trusses and the errors of calculation are greater. The same general methods of calculation are used

as in the double systems already discussed. In getting web stresses the chord members are assumed as straight between successive joints of the same system.

Section VI.—Trusses with Subdivided Panels

106. The Baltimore Truss.—The Baltimore truss, Fig. 78, or a modified form of it, Fig. 85, is used very generally for long spans. The stresses are all easily determined; the panel lengths are short, while the diagonals have an economical inclination. The intermediate floor beams at b, d, etc., are carried by means of the subverticals, $b\,b'$, $d\,d'$, etc., to the joints b', d', etc., at the centre of the main diagonals. These points are in turn supported by the substruts $b'\,c$, $d'\,c$, $f'\,e$, etc. The dotted members, $f'\,G$ and $h'\,I$, are counters. When they come into

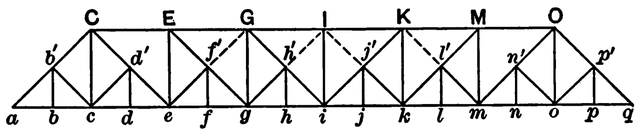


FIG. 78.

action the substruts, $e\,f'$ and $g\,h'$, act with them and become tension members. Fig. 84 shows a form in which the centre joints are supported by subties extending to the upper chord joints. The stresses in the various members will be discussed in detail.

Chord Stresses.—The upper chord stresses are found as usual by taking centres of moments at lower chord points, the counters being omitted. Thus, in Fig. 79, $S_1 = M_e \div h$.

The stresses in the lower chord members are found by taking moments about upper chord points. Thus for piece $d\,e$ pass a section cutting $C\,E$, $d'\,e$, and $d\,e$, Fig. 79, and write Σ mom. about $C = 0$. We have, therefore, $R_1 \times 2\,d - P_1 \times d + P_3 \times d - S_3 \times h = 0$, whence S_3 is determined. It is to be noted that the moment of the external forces acting on the portion considered, about the point C, is not the usual "bending moment" in the truss at C, since in this case we have included the force P_3 which is on the *right* of the centre of moments. In the equilibrium polygon this moment is represented by the ordinate $c'\,c''$.

The stress in $c\,d$ is evidently equal to that in $d\,e$. The other chord stresses are found in a similar manner to that described above.

Web Stresses.—The stress in each subvertical, $b\,b'$, $d\,d'$, etc., is equal to the load at its base $= (W + P)$. The vertical component of the compressive stress in each of the pieces, $b'\,c$, $d'\,c$, etc., is equal to one-half the stress in the subvertical, as is easily shown. Thus in Fig. 80, showing the forces acting on the joint d', let the stress S in $c\,d'$, be replaced by its components S_v and S_h applied at c. An equation of moments about e gives $S_1\,d = S_v \times 2\,d$ whence $S_v = \frac{1}{2}\,S_1$.

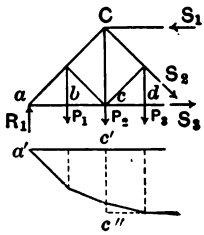


FIG. 79.

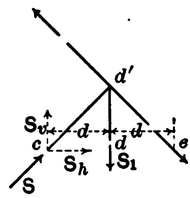


FIG. 80.

The maximum stresses in the members $a\,b'$, $d'\,e$, $f'\,g$, and $h'\,i$ are determined from the maximum positive shears in the respective panels, as these members carry the entire shear. The maximum stresses in the counters $I\,j'$ and $K\,l'$ are likewise found from maximum positive shears in these panels. The stresses in the verticals $E\,e$, $G\,g$, and $I\,i$ are equal to the vertical components in $E\,f'$, $G\,h'$, and $I\,j'$, respectively, plus whatever load may be applied at the upper panel point. The tension in $C\,c =$ vert. comp. in $b'\,c +$ vert. comp. in $d'\,c +$ load at $c = 2\,(W + P)$.

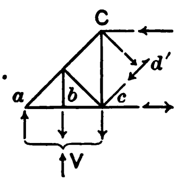


FIG. 81.

The stresses in the upper portions of the main diagonals, $b'\,C$, $E\,f'$, etc., require special attention. Consider the member $C\,d'$, passing a section through the panel $c\,d$, Fig. 81. In general, for positive shear, we have the relation: Vert. comp. $C\,d' = V -$ vert. comp. $c\,d'$. This enables the dead-load stresses to be readily determined. For maximum live-load stress the position of loads must first be found. Consider the effect of a joint load, P, placed at d. This load adds to the reaction and to the shear on the section an amount equal to $13/16\,P$. It also

adds to the vertical component of the stress in $c\,d'$ an amount equal to $\frac{1}{2}\,P$. The net effect on the vert. comp. in $C\,d'$ is therefore $(13/16 - \frac{1}{2})\,P = 5/16\,P$. This shows that the live load should extend from the right up to joint d. The vertical component of the live-load stress in $C\,d'$ is then equal to the shear minus $\frac{1}{2}\,P = 1/16\,P\,(1 + 2 + \ldots + 13) - \frac{1}{2}\,P = 83/16\,P$. Similarly for $E\,f'$ and $G\,h'$. The vertical component in $b'\,C$ = shear in $b\,c$ + vert. comp. $b'\,c$, as is seen from Fig. 82, from Σ vert. comp. = 0. Adding P to joint b decreases the positive shear in $b\,c$ by $1/16\,P$, but increases the vertical component in $b'\,c$ by $8/16\,P$, thus increasing the vertical component in $b'\,c$ by $7/16\,P$. Hence for a maximum stress in $b'\,C$ the bridge should be fully loaded.

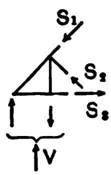


Fig. 82.

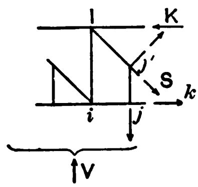


Fig. 83.

When the pieces $j'\,k$ and $l'\,m$ (and similar members on the left), act with the counters, they are in tension, and their maximum stresses will occur when the shear in these panels is positive (truss loaded on the right). Consider member $j'\,k$. When the shear in $j\;k$ is positive the shear in panel $i\,j$ is also positive and the active diagonal in the panel will be $I\,j'$ (both $I\,j'$ and $i\,j'$ being tension members). The conditions will therefore be as represented in Fig. 83. Whatever load, P, there may be at j produces a stress in the member $j'\,K$ whose vertical component is equal to $\frac{1}{2}\,P$. Therefore in general the vertical component of the tension in $j'\,k = V$ + vert. comp. $j'\,K = V + \frac{1}{2}\,P$. A process of reasoning similar to that employed from member $d'\,e$ shows that for a maximum tension in $j'\,k$ the live load should extend only to joint k. The vert. comp. of maximum live-load stress, therefore $= 1/16\,P\,(1 + 2 + \ldots + 6)$; the corresponding dead-load stress $= V + \frac{1}{2}\,W = -1\frac{1}{2}\,W + \frac{1}{2}\,W = -W$. This stress is compressive. The stress in $l'\,m$ is found in like manner. If the total shear in $l\,m$ plus $\frac{1}{2}\,W$ (= vert. comp. in $l'\,m$) should be negative, then there would be no tensile stress in $l'\,m$.

The minimum stresses in subverticals, substruts, and hip-verticals are the dead-load stresses. The minimum stresses in all other web members up to the first counter, $f'\,G$, are caused by loading those joints which were not loaded for maximum stresses. The stresses themselves are calculated as before. In the counters, and in members $f'\,g$ and $h'\,i$, the minimum is zero. In members $G\,h'$ and $G\,g$ the minimum occurs when the truss is loaded from the left end up to g, or far enough to bring counter $h'\,I$ into action, but without placing a load at h. The piece $h'\,i$ is then not acting and the vert. comp. in $G\,h'$ will equal one-half the dead joint load at h. The stress in $G\,g$ will then equal the vert. comp. in $G\,h'$ plus the dead load applied at G. The minimum stress in $I\,i$ will be the dead load at I.

Fig. 84 illustrates a form of Baltimore truss in which the loads at the intermediate joint are carried by subties to the upper chord joints. A substrut is required at b'. Member $g\,h'$ is a counter.

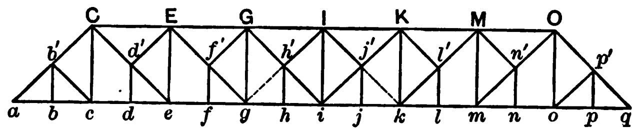


FIG. 84.

The analysis of this truss is very similar to that of the other form, and a brief statement with respect to certain members will suffice. The maximum stresses in members $C\,d'$, $E\,f'$, and $G\,h'$, and in $I\,j'$, when acting as a counter, are due to the maximum positive shears in the respective panels. The minimum stresses in $C\,d'$ and $E\,f'$ are found from the minimum shears, but in $G\,h'$, and in all the subties, the minimum stress is equal to one-half a lower dead joint load $\times$ sec θ. The counter $g\,h'$ will have the same maximum tension as the member $g\,h'$ or $j'\,k$ of Fig. 78; its minimum stress is zero. The vert. comp. in $d'\,e$ is equal to the shear in the panel $d\,e$ plus the vert. comp. in $d'\,E$ (one-half the load at d). The maximum stress in $d'\,e$ will occur when the truss is loaded from the right up to d, and its minimum stress when joints b and c are loaded. The maximum stresses in $f'\,g$ and $h'\,i$ are similarly found. The minimum stress in $h'\,i$ is zero, and likewise in $f'\,g$ if the shear in this panel can be negative. If not, then its minimum stress is found as for $d'\,e$.

The stress in $E\,e$ is equal to the sum of the vert. comp. of the stresses in $E\,f'$ and $E\,d'$, plus the load at E. Its maximum value occurs when the truss is loaded from the right up to d; its minimum value when the other joints are loaded. The maximum in $G\,g$ and $I\,i$ are similarly found, except that in these cases the bridge should be loaded up to h and j respectively. The minimum stress in $G\,g$ occurs when the bridge is loaded on the left sufficiently to throw $h'\,i$ out of action. Its stress is then equal to vert. comp. $G\,h'$ + vert. comp. $f'\,G$ + load at G = lower dead joint load + upper dead joint load. The minimum stress in $I\,i$ is the same.

107. The Pettit Truss.—The Pettit truss, shown in Fig. 85, is the standard form for very long spans. It is very similar to the Baltimore truss, the only difference being in the inclined upper chord, which is a more economical arrangement for long spans. Usually, also, the sub-

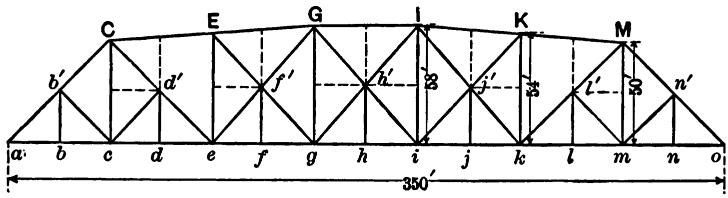


FIG. 85.

paneling does not extend to the end of the bridge, as the height at that point does not require it (see Fig. 86). The pieces shown by dotted lines serve merely to support the chords and posts at intermediate points and form no part of the truss proper. The vertical ones are designed to carry the weight of the upper chord; the horizontal ones have no definite load and are made of uniform size, sufficiently strong to resist the buckling of the posts. Omitting these members, the analysis offers no special difficulties, as the variation from the Baltimore truss due to inclined chords is easily taken into account.

Fig. 85 shows the use of substruts. The more common arrangement, however, is to use subties running to the upper chord joints as shown in Fig. 86. In this form the inclination of $d'\,E$ is not equal to that of $C\,d'$, and it follows that the vertical component in $d'\,E$ is not equal to one-half the load at d. The value of this stress can readily be found by a diagram of joint d', but algebraically it is best found by the

use of a moment equation. Let Fig. 87 represent the forces acting on the joint d', and let S be the desired stress. Let P = stress in $d'\,d$ = load at d. Replace S by its components applied at point E, and take moments about e. We have at once

$$S_h = \frac{P\,d}{h} \quad \text{and} \quad S = S_h\frac{l}{d} = \frac{P\,l}{h},$$

where l = length of $d'\,E$ and d = panel length. This easily obtained result is a good illustration of the usefulness of the moment equation for concurrent forces and of the device of substituting for an inclined force its components applied at a convenient point.

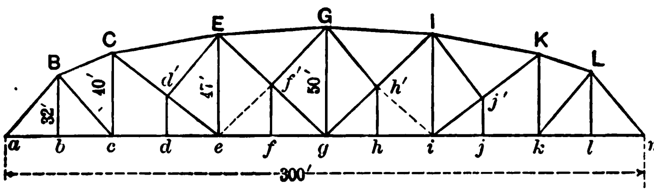


FIG. 86.

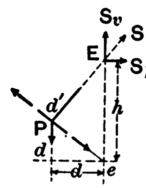


FIG. 87.

EXAMPLE.—To illustrate the more difficult parts of the anaylsis the maximum stresses will be found in members $d'\,e$, $E\,e$, $E\,f'$ and $e\,f'$ of Fig. 86. The dead load will be taken at 1,500 lbs. per ft. per truss and the live load at 2,400 lbs. per ft. per truss. All the dead load will at first be assumed as applied at the lower chord joints. The live joint load = P = 60,000 and the dead joint load = W = 37,500. Length of $d'\,E$ = $\sqrt{25^2 + 27^2} = 36.80$; length of $E\,f' = e\,f' = \sqrt{25^2 + 23.5^2} = 34.31$.

Stress in $E\,f'$.—The live load extends from the right up to f and $e\,f'$ is not in action. The stress in $E\,f'$ will be obtained from the shear in panel $e\,f$.

The calculations are briefly given.

Dead-load stress. Shear = $1.5\,W$. $R_1 = \frac{11}{2}W$. H. comp. $E\,G = \left[\frac{11}{2}W \times 6 - W(5+4+3+2)\right]\frac{d}{50} = 19\,W.\frac{d}{50}$. V. comp. $E\,G$ = H. comp. $E\,G \times \frac{3}{50} = 0.57\,W$. V. comp. $E\,f' = (1.5 - .57)\,W = 34{,}870$ lbs. tens.

Live-load stress. Shear $= \frac{P}{12}(1 + 2 + \ldots + 7) = \frac{7}{3}P$. V. comp. $E\,G = \frac{3}{50} \times \left(\frac{7}{3}P \times 6\right)\frac{d}{50} = 0.42\,P$. V. comp. $E\,f' = \left(\frac{7}{3} - .42\right)P = 114{,}800$ lbs. tens.

Stress in $E\,e$. Loads same as for $E\,f'$.

Dead-load stress. V. comp. $d'\,E$ = H. comp. $d'\,E \times \frac{27}{25} = W.\frac{d}{47} \times \frac{27}{25} = 0.574\,W$.

V. comp. $C\,E = \frac{7}{50}\left[\frac{11}{2}W \times 4 - W(3+2)\right] \times \frac{d}{47} = 1.266\,W$. Stress in $E\,e$ =

shear in panel $e\,f$ + V. comp. $d'\,E$ − V. comp. $C\,E = (1.5 + .574 - 1.266)\,W =$ 30,300 lbs. comp.

Live-load stress. V. comp. $C\,E = \frac{7}{50} \cdot \left(\frac{7}{3}\,P \times 4\right)\frac{d}{47} = 0.695\,P$. Stress in $E\,e =$ $\left(\frac{7}{3} - .695\right)P =$ 98,300 lbs. comp.

Stress in $d'\,E$. This stress is found from loads $W + P$ at d. Consider joint d' as in Fig. 87.

Dead-load stress $= W \times \frac{36.8}{47} =$ 29,360 lbs. tens. Live-load stress $= P \times \frac{36.8}{47} =$ 46,980 lbs. tens.

Stress in $e\,f'$. Load joints a to e. Member $f'\,g$ goes out of action. Stress in $f'\,G$ will be obtained and thence the stress in $e\,f'$ by considering joint f'.

Dead-load stress. Member $f'\,G$, produced, intersects the vertical $E\,e$ at a point 3 feet below e. This intersection may be taken as moment center for stress in $E\,G$. Stress in $f\,g$ (moments at G) $= 9\,W$.

Then V. comp. $E\,G = \frac{3}{50}\left[\frac{11}{2}\,W \times 4d - W(3+2+1)d + W\,d + 9\,W \times 3\right]\frac{1}{50} = 0.54\,W.$

Shear in $f\,g = 0.5\,W$. V. comp. $f'\,G = (0.5 - 0.54)\,W = -0.04\,W$, indicating tensile stress. Then with forces acting at f' and with moment centre at E we have

$$\text{H. comp. } e\,f' = \frac{W \times 25 - \text{H. comp. } f'\,G \times 50}{47} = \frac{W\left(25 - 0.04 \times \frac{25}{26.5} \times 50\right)}{47} = \frac{23}{47}\,W.$$

Stress in $e\,f'$ = H. comp. $\times \frac{34.31}{25} = 0.672\,W =$ 25,200 lbs. comp.

Live-load stress. Shear in $f\,g = \frac{P}{12}(1 + 2 + 3 + 4) = \frac{10}{12}\,P$.

Stress in $f\,g = \frac{10}{12}\,P \times 6 \times \frac{25}{50} = 2.5\,P$. Then V. comp. $E\,G = \frac{3}{50}\left(\frac{10}{12}\,P \times 8\,d + 2.5\,P \times 3\right)\frac{1}{50} = 0.209\,P$. V. comp. $f'\,G = \left(\frac{10}{12} + 0.209\right)P = 1.042\,P =$ 62,500 lbs.

H. comp. $e\,f =$ 62,500 $\times \frac{25}{26.5} \times \frac{50}{47} =$ 62,800 lbs., and stress in $e\,f =$ 62,800 $\times \frac{34.31}{25} =$ 86,200 lbs. tens. The net tension in $e\,f' =$ 86,200 − 25,200 = 61,000 lbs.

108. The Compound-Triangular Truss.—Fig. 88 shows a method of subdividing the panels in the double-triangular truss. The computation of stress is but slightly affected by the sub-panelling.

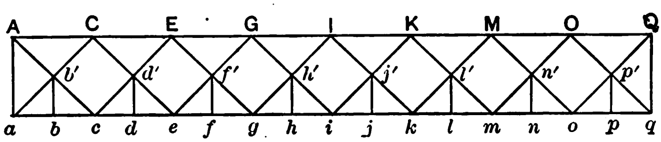


FIG. 88.

The intermediate joint loads at b, d, f, etc., may be considered as transferred to the main panel points a, c, e, etc., by means of separate

small trusses or trussed stringers, $a\,b'\,c$, $c\,d'\,e$, etc., as shown in Fig. 89. Whatever stresses may exist in the inclined and horizontal members of these small trusses must be added to the stresses for the same loading in those members of the main truss with which they in reality coincide.

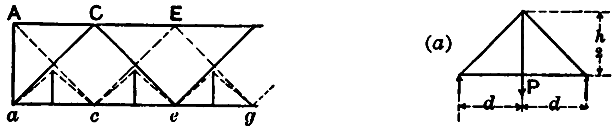


FIG. 89.

To find the chord stresses in such a truss, neglect, at first, the subpanelling and calculate the chord stresses for the main truss as explained in Art. 103, assuming each of the joints c, e, g, etc., loaded with a double panel load. The resulting stresses will be correct for the upper chord. For the lower chord there will need to be added to each stress thus found the stress in the member as a part of the small truss. This additional stress [see Fig. 89 (a)] $= P\,d/h$, where P is the joint load (dead or live), applied at b, d, f, etc.

The dead-load web stresses are also found by considering at first the main truss only. The resulting stresses will be correct for the upper halves of the web members. For the lower halves, $a\,b'$, $b'\,c$, $c\,d'$, etc., a compression must be added equal to $\frac{1}{2}\,W \sec\theta$ [Fig. (a),] due to the action of the small truss.

The maximum live-load stresses in the upper halves of the diagonals can also be found by omitting the subpanels. The maximum tension

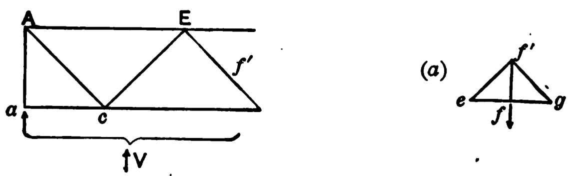


FIG. 90.

in $E\,f'$, for example, occurs when the main joints, g, k, and o, are loaded with a double panel load; the maximum compression occurs when joint c is so loaded. For the lower halves of these members the stresses in the small trusses must be considered in detail. Consider member

$f'g$ for example, belonging to the dotted web system (Fig. 90). Its vert. comp. is always equal to the shear on the section cutting $E g$ of the main truss, plus a compression equal to one-half the live load at f, if any. The positive shear on the section is greatest when joints g, k, and o are fully loaded, but to load g fully requires a load at f, which would cause a compression in $f'g$ of the small truss. The resultant effect of a joint load, P, placed at f must be determined. Adding such a load at f adds $\frac{1}{2} P$ at g, thereby increasing the shear by $5/8 \times 1/2\,P = 5/16\,P$. The vert. comp. of the compression caused in $f'g$ of the small truss $= 8/16\,P$, hence the resultant effect is compressive by $3/16\,P$. Hence for maximum tension no load should be placed at f. Joint g will then be loaded with $1\text{-}1/2\,P$, and joints k and o with $2\,P$ each, giving a shear equal to $1/8 \times 2\,P + 3/8 \times 2\,P + 5/8 \times 1\frac{1}{2}\,P = 31/16\,P$, which is the vert. comp. of the desired maximum tension. For the maximum compression in $e f'$, the joint f should be loaded. On the full system of the main truss there will then be $1/2\,P$ at e and $2\,P$ at joints i and m. The shear in panel $e g$ will equal $2/8 \times 2\,P + 4/8 \times 2\,P + 6/8 \times 1/2\,P - 1/2\,P = 11/8\,P$. The vert. comp. of the compression in $e f$ is therefore $11/8\,P + 1/2\,P = 15/8\,P$. The minimum stresses are caused when those joints are loaded which were not loaded for maximum stress.

In a similar manner the maximum live-load stress in $h' i$ is found to require the load to extend to i, giving loads on the main joints of $1\text{-}1/2\,P$ at i and $2\,P$ at m. The vertical component of the tension in $h' i' = 5/4\,P$. The vertical component of the maximum compression in $g h' = (1/8 \times 2\,P + 3/8 \times 2\,P + 5/8 \times 1/2\,P - 1/2\,P) + 1/2\,P = 21/16\,P$, loads extending to h.

SECTION VII.—SKEW-BRIDGES

109. Skew-bridges are those in which one or both of the end-supports of one truss are not directly opposite those of the other. Fig. 92 is a plan and Figs. 91 and 93 are elevations of the two trusses of such a bridge. The intermediate panel points are usually placed opposite in the two trusses, so that all floor beams are at right angles to the line of the truss. Where the skew is not exactly one panel, as at the left end, it is desirable to shorten the panel $B\,C$ and lengthen $B'\,C'$ in order

that the figure *a B B′ b* may be a plane figure. The hip-verticals are thus made inclined.

In the analysis, each truss must be treated separately unless the skew is the same at each end and the trusses therefore alike. In calculating the joint loads on the trusses the floor load may be assumed as applied along the centre line X *Y*. Each joint load is equal to the

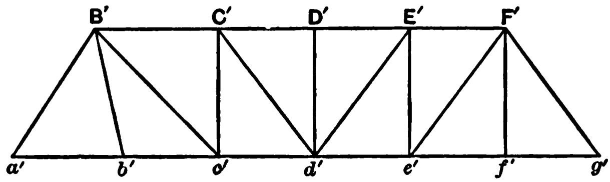


FIG. 91.

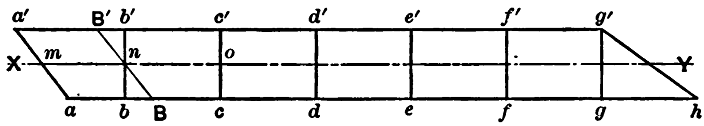


FIG. 92.

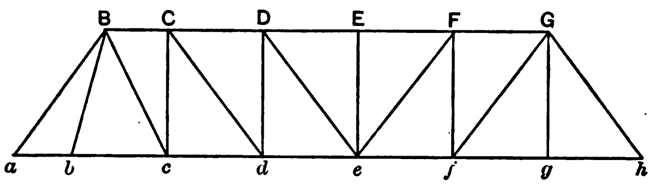


FIG. 93.

usual full panel load excepting the loads at *g*, *b* and *b′*. At *g* it is three-fourths of a full joint load and at *b* and *b′* they are each equal to ¼ × (load on panel *m n*) + ¼ × (load on panel *n o*). The joint loads being known the stresses are readily calculated by the principles already discussed. For maximum live-load stresses the position of the moving load is in general the same as for the square bridge.

CHAPTER V

ANALYSIS OF BRIDGE TRUSSES FOR CONCENTRATED LOADS

110. The preceding chapter has treated all live loads as uniform loads. While this method of treatment is in general use for highway bridges and, to some extent, for railway bridges, it is the general practice in the latter case to deal with actual specified wheel loads and to find the maximum stress in each member due to these loads. In highway bridges, also, concentrated loads generally need to be considered in determining the maximum stresses in the floor system. It is proposed in the following discussion to show how to find the position of any given system of loads which will produce the maximum stress at any section of a beam or in any truss member and to explain methods of calculating such stress. The methods apply in general to any system of moving loads and hence apply also to the uniform load system treated in the preceding chapter.

SECTION I.—INFLUENCE LINES

111. Definition and Construction.—Before proceeding with the consideration of particular structures a general explanation will be given of a graphic method of representing the variation in moment, shear, stress, or other function, relative to a particular section or member of a structure, which is of very general application in all problems dealing with moving loads. Suppose, for example, that the bending moment at C, in the beam $A\,B$, is under investigation (Fig. 1). We will first trace the variation which occurs in this bending moment when a single concentrated load, P, passes over the beam. Suppose this load to move across the beam from B to A. So long as it is between B and C the bending moment at C is equal to $P\frac{x}{l}.a$. This moment

varies directly with x and may therefore be represented by the straight line $B' D$, drawn with reference to a horizontal axis $A' B'$ so that the ordinate y at any point $= P\frac{x}{l}.a =$ the moment at C for load P placed at a distance x from B. The ordinate $C' D$, under the point C, will be the moment at C for load P at C, $= P.a\,a'/l$. After the load passes C the bending moment at C (considering forces on the right of C), is equal to $Pa'x'/l'$. This moment is represented by the straight line DA'. The ordinates to the entire broken line $A'DB'$ therefore represent the moment at C due to the concentrated load P as it moves over the beam. Generally, for convenience, the load P is taken equal to unity and the resulting line will therefore represent the effect of a unit load moving over the structure. By reason of its general significance such a line is called an *influence line.*

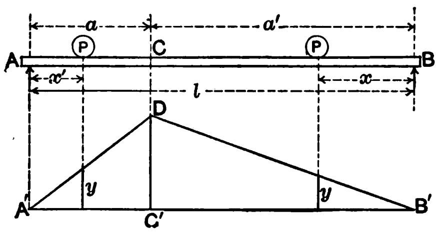


FIG. 1.

An influence line may then be defined in general as a line which represents the variation of moment, shear, panel load, stress, or any similar function, at a particular point in a structure or in any particular member, due to a load unity moving over the structure. The value of the function for any given position of the unit load is measured by the ordinate to the influence line *at the point where the load is placed.* An influence line differs from the usual moment or shear curve heretofore employed in that the former represents the variation in the function for a *particular point,* due to a moving load, while the latter represents the variation in the function *along the structure* due to a fixed load.

The equation of the influence line for any function may be derived by writing out the value of the function for a load unity when placed at a variable distance x from one end of the structure taken as the origin. Influence lines for simple structures are composed of one or more straight lines, as the functions are all of the first degree; and in most of the structures considered in this chapter the influence lines are readily constructed by calculating the ordinates at one or two critical points.

112. Influence Lines Between Points of Application of Loads.—In any jointed structure or girder, where the applied load is transferred by members of a floor system (stringers, etc.), to certain load points, the influence line between consecutive load points is always a straight line. This is proven as follows: Let y_1 and y_2 (Fig. 2), be the ordinates of the influence line when the load is placed at points 1 and 2 respectively. For intermediate positions the proportion of the load P carried to adjacent points, 1 and 2, will be $P\ x/d$ and $P\ x'/d$, respectively, and will vary uniformly as P moves from 1 and 2. The ordinate

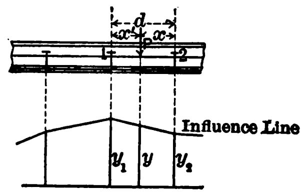


FIG. 2.

y will be equal to $P\dfrac{x}{d}\cdot y_1 + P\dfrac{x'}{d}\ y_2$, and this quantity will also vary uniformly between the values y_1 and y_2. Hence the influence line for the panel 1–2 is a straight line. The influence line for any such structure can therefore always be completely determined by calculating the value of the function for a load unity placed successively at each load point.

113. Use of Influence Lines.—Influence lines are particularly useful in representing graphically the influence upon the value of any function of the loads on the different parts of a structure and the effect of moving those loads in either direction. In structures whose analysis is difficult they are also very useful in the actual calculation of stresses.

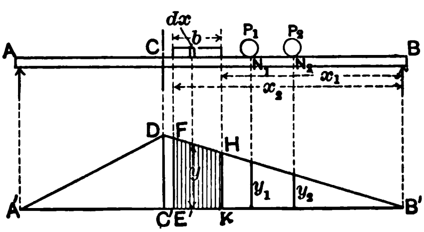


FIG. 3.

In Fig. 3 suppose $A'D\,B'$ be the influence line for bending moment at C in the beam $A\ B$, constructed for unit load as explained in Art. 111. Then the bending moment at C due to a unit load placed at any point, N_1, is given by the corresponding ordinate y_1. A load of P_1 placed at point N_1 will cause a moment P_1 times as great, and hence equal to $P_1\,y_1$. Likewise a load P_2 at N_2 will cause a moment at C equal to $P_2\,y_2$, etc. Hence, in general, the total bending moment at C due to any number of concentrated loads is equal to the sum of

the products of the loads times the respective ordinates of the influence line or

$$M = \Sigma P y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This principle applies to any function under consideration.

Again, if the load consists of a uniform load of p per unit length, and of any length, b, the total moment due to this load is equal to the load p per unit length, multiplied by the area within the limiting ordinates of the influence line. For the moment due to an element of load, $p\,d\,x$, is $p\,d\,x\,y$, where y is the ordinate under the load; and integrating between the limiting values of x (x_1 and x_2), we have:

$$M = p \int_{x_1}^{x_2} y\,d\,x = p \times \text{area } E\,F\,H\,K.$$

From this explanation it is clear how an influence line can readily be used for direct calculation of the desired quantities for any system of loading.

In the following articles, in which certain moments, shears, etc., are investigated, the influence line is first drawn, and from this is derived the general rule or criterion for the position of a series of concentrated loads which will cause the value of such function to be a maximum. The direct algebraic method of deriving such criterion is also given in some cases. It is a briefer method for the simpler problems, but the use of influence lines leads to a better understanding of the effect of moving loads and is almost necessary in the case of the more complex structures.

Section II.—Beams Loaded Continuously

114. **Reaction and End Shear.**—*Influence Line.*—The left reaction, or end shear, of the beam $A\,B$, Fig. 4, due to a unit load placed at any point on the beam, $= 1 \times x/l$. The influence line for reaction is then the straight line $A''\,B'$, drawn with ordinate at A' equal to unity. The influence line for the right reaction may be drawn below the axis and is the line $A'\,B''$. The right reaction will not be considered further.

115. *Position of Loads for Maximum Values.*—For a uniform load the maximum reaction evidently occurs when the entire span is loaded and is equal to $p \times$ area $A'\,B'\,A'' = \frac{1}{2}\,p\,l$. For a single concentrated

load P_1, moving over the structure, the left reaction is equal to $P_1 y_1$; and this reaches a maximum value when the load reaches the end where it equals P_1. For a series of concentrated loads P_1, P_2, P_3, etc., the total reaction is equal to $P_1 y_1 + P_2 y_2 + P_3 y_3 +$, etc. As the loads move towards the left each of these terms increases in value until the first load, P_1, passes off the beam, when the reaction is reduced suddenly by the value of this load. The reaction then again increases until P_2 passes off, and so on. It therefore reaches a maximum each time a load reaches A. Which of these maximum values is the greatest depends upon the weight and spacing of the several loads; it can readily be determined by trial.

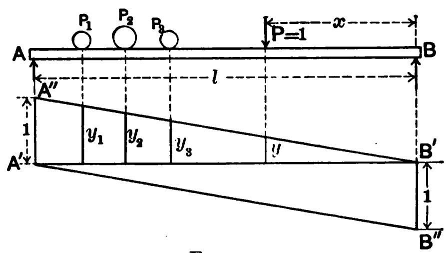


FIG. 4.

116. *Calculation of Maximum Reaction.*—In calculating the value of maximum reaction for a series of concentrations, the series should be placed with the heavier loads near the left end and one of the loads at the end. The reaction is readily found by the usual moment equation about B. Various positions may need to be tried and the greatest value taken. Convenient tabular forms for such calculations are explained further on. A graphical method of calculation is given in the following article.

117. The Reaction Polygon.—In Fig. 5 let P_1, P_2, P_3, and P_4 be a series of loads moving from B towards A. The reaction due to P_1 alone may be represented as in Art. 74, Chap. IV., by the line $A_1 B'$, the end ordinate being equal to P_1. When P_1 passes the point N_1, whose distance from B is equal to b_1, then P_2 comes upon the beam and the increase in reaction is now due to both P_1 and P_2. The additional effect of P_2 is represented above the line $B' A_1$ by the line $B_1 A_2$, drawn so that $A_1 A_2$ is equal to the reaction due to P_2 when P_1 is at A, $= P_2 \dfrac{l - b_1}{l}$. The ordinate between these two lines at a distance b_1 to the left of the reaction will be P_2. The total ordinate from $A' B'$ to the line $A_2 B'$ will now represent the reaction due to P_1 and P_2. In a similar manner the effect of P_3 is added to that of P_1 and P_2, and the

same construction can be continued for any number of loads until P_1 passes off the beam. The reaction due to any given position of the loads is then represented by the total ordinate y under load P_1, when placed in the given position. The broken line $A_4 B_3 B_2 B_1 B'$ may be called a *reaction polygon* for the given series of loads, and for the beam $A B$.

When P_1 passes off the beam then P_2 will head the series, and the line $A_1 B'$ may be assumed as the base line and the beam shifted a distance b_1 towards the left so that the end A comes directly over the end C_1 of this base line. The construction can then be continued and the reaction will now be given by ordinates from the line $C_1 B_1$.

Finally, it may be stated in general that the maximum reaction for the entire series of loads will be given by the maximum ordinate be-

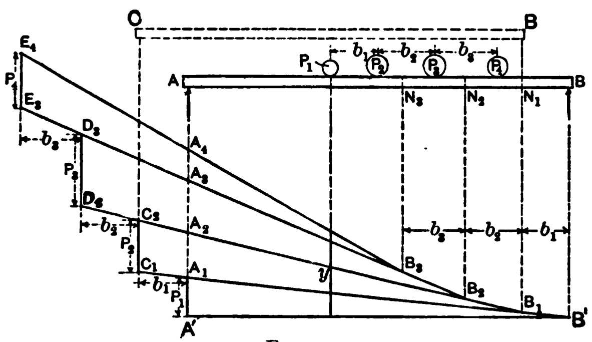


FIG. 5.

tween the base of the diagram formed by the lines $B' A' A_1 C_1 C_2 D_2$, etc., and the reaction polygon.

If a moving uniform load be considered the reaction polygon becomes a parabola, as given in Art. 74, which is also the maximum shear curve.

118. Shear at any Point in a Beam.—*Influence Line.*—The shear at C in the beam $A B$, Fig. 6, due to a load unity moving from B towards A, increases from zero when the load is at B to $+\dfrac{l-a}{l}$ when the load is just to the right of C. As the load passes C the shear becomes $-a/l$ and then increases to zero when the load reaches A. The lines $B' D$ and $A' E$ are parallel and the ordinates at A' and B' are equal to unity.

119. *Position of Loads for Maximum Shear.—(a) Uniform Loads.*—The maximum positive shear due to a moving uniform load of p per unit length will evidently occur when the load extends from B to C and is equal to $p \times$ area $D\,C'\,B' = p \times \frac{1}{2}\,\frac{l-a}{l}\,(l-a) = \frac{P}{2l}(l-a)^2$, as in eq. (8), Art. 74. The maximum negative shear requires $A\,C$ to be loaded, giving a value equal to $-\frac{P}{2l}a^2$ as in eq. (9).

(*b*) *Concentrated Loads.*—Suppose the beam to be loaded in any manner with a series of loads, Fig. 7. The influence line shows that all loads to the right of C cause positive shear and all loads to the left, negative shear. Consider now the effect of a movement of the series towards the left. The positive shear due to all the loads on the right will be increased and the negative shear due to those on the left will be decreased, hence in all cases the positive shear will be increased by such movement until some load P_1 passes point C, when the shear will suddenly be decreased by an amount equal to $P_1 \times D\,E = P_1$. For concentrated loads therefore the shear reaches a maximum value each time a load reaches C. The greatest of these maximum values is the one sought; it will evidently occur when there are few or no loads to the left of C.

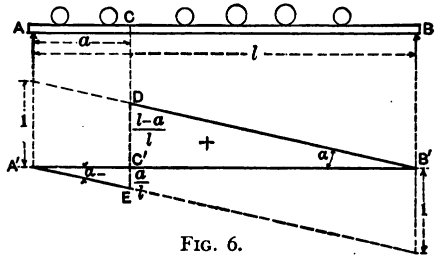


FIG. 6.

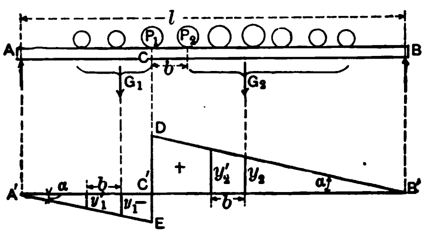


FIG. 7.

It will be of some assistance to determine the relative value of the shears when each of two consecutive wheels, P_1 and P_2, are placed just to the right of C (Fig. 7). Let G_1 represent the sum of the loads to the left of and including P_1, and G_2 the sum of the loads to the right of P_1. Let G = total load on the beam when P_1 is at C, and b = distance between P_1 and P_2. Suppose the loads advance from this position

a distance b, thus bringing P_2 at C. The effect upon the positive shear is first to decrease it suddenly by an amount P_1, after which it is gradually increased. The increase due to G_2 may be expressed by $G_2 \times (y_2' - y_2) = G_2\, b \tan a$, and the increase due to G_1 (decrease in negative shear), may likewise be expressed by $G_1\ (y_1' - y_1) = G_1\, b \tan a$. Hence the total increase is $G\, b \tan a = G\, b/l$; and the net change in shear due to the entire movement $= G\, b/l - P_1$. If this expression is positive then the second position gives the greater shear and, if negative, the first position. For equal shears we have therefore,

$$\frac{G}{l} = \frac{P_1}{b}. \qquad (1)$$

The slight increase in shear due to additional loads that may come upon the beam from the right has been neglected. If G' be the total load on the beam when P_2 is at C, then the increase in shear will be somewhere between $G\, b/l - P_1$ and $G'\, b/l - P_1$. When the former expression is negative and the latter positive, then both positions should be tried.

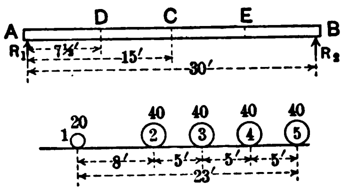


FIG. 8.

If the wheels are all equal and uniformly spaced, then P_1/b will always be greater than G/l, and the shear will be a maximum when the first wheel of the series is placed at the point. If, however, the first wheel is small, or the first space, b, large, then it may easily happen that the second wheel should be placed at the point.

If the end section at A be considered, the loads immediately pass off the beam when passing the section. Hence in comparing the values of the shear the value of G in eq. (1) should not include the load P_1.

120. *Calculation of Shears.*—To illustrate the application of Art. 119, the maximum positive shears will be calculated at points A, D and C of the beam $A\ B$, (Fig. 8), for the series of loads shown.

Shear at A.—Consider the relative shears with P_1 and P_2 placed at A. By eq. (1) $G/l = 160/30 = 5.3$ and $P_1/b = 20/8 = 2.5$, hence the shear is greater when P_2 is at A. Consider P_2 and P_3. Here $G/l = 120/30 = 4.0$ and $P_2/b = 40/5 = 8$, hence P_2 placed at A

gives a maximum. The value of the shear $= R_1 = 40\ (30 + 25 + 20 + 15) \div 30 = 120$.

Shear at *D*.—Consider P_1 at *D*, then P_2. The value of *G* as the loads move 8 feet towards the left varies from 140 to 160. In eq. (1) G/l then varies from 4.6 to 5.3, and $P_1/b = 20/8 = 2.5$. Hence P_2 should be placed at *D*. The shear $= 40\ (22.5 + 17.5 + 12.5 + 7.5) \div 30 = 80$.

Shear at *C*.—Considering P_1 and then P_2 at *C*, the value of *G* varies from 100 to 140 and G/l from 3.3 to 4.3. Again, $P_1/b = 2.5$ and the shear is a maximum with P_2 at *C*. Shear $= R_1 - P_1 = [20 \times 23 + 40\ (15 + 10 + 5)] \div 30 - 20 = 35.3$.

Shear at *E*.—For P_1 and P_2 at *E*, *G* varies from 20 to 100 and G/l from 0.67 to 3.3. $P_1/b = 2.5$. Both positions will therefore be tried. With P_1 at *E*, shear $= R_1 = 20 \times 7.5 \div 30 = 5.0$. With P_2 at *E*, $R_1 = [20 \times 15.5 + 40\ (7.5 + 2.5)] \div 30 = 23.7$, and shear $= 23.7 - 20 = 3.7$. The greater shear in this case therefore is for P_1 at *E*.

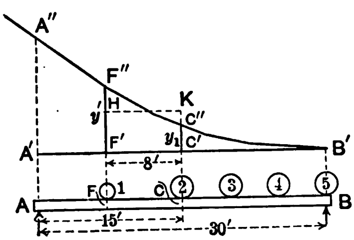


FIG. 9.

121. *Use of Reaction Polygon in Calculating Shears.*—In Art. 117 it was shown how the reaction polygon may be used for calculating end shears. It may also be employed in getting shears at other sections. In Fig. 9, suppose $B'\,A''$ is the reaction polygon constructed for the beam and loads of Fig. 8. The ordinate to this line at any point will be the left reaction when the loads are placed with P_1 at the given point. The reaction polygon therefore gives the shears throughout the beam for the case when the loads are placed with P_1 just to the right of the given section. As seen in the above analysis this will be the maximum shear for sections towards the right end. Consider the section at *C*. The shear at *C* for P_1 placed at this point (just to the right) is given by the ordinate y_1. If the loads are now placed with P_2 at *C*, as in the figure, P_1 is at *F*, 8 feet to the left of *C*, and the left reaction is given by the ordinate y'. The shear between *F* and *C* will equal $y' - P_1$, and is given by the ordinate to the horizontal line $H\,K$ drawn with $F''\,H = P_1$. In this case $K\,C'$ is greater than $C''\,C'$, showing that the shear is

M_c - $P_1 a$

M_c - $R_1 a'$ - $\frac{x x'}{l}$

greater when P_2 is placed at the point. In a similar manner the shear can be found for P_3 at the point. If some of the loads pass off the beam then the base line of the reaction polygon is to be taken as indicated in Art. 117. Notice that the slope of a line drawn from F'' to K is P_1/b of eq. (1), and the slope of a line $F''\ C''$ is G/l, thus giving a graphical representation of the relation expressed in eq. (1).

122. Bending Moment in a Beam.—*Influence Line.*—In Fig. 10, the influence line for bending moment at any point C, distance a from A, is $A'\ D\ B'$, drawn as explained in Art. 111. The ordinate $C'\ D = a\,a'/l$, and the ordinates at A and B, to the lines produced, are equal to a and a' respectively.

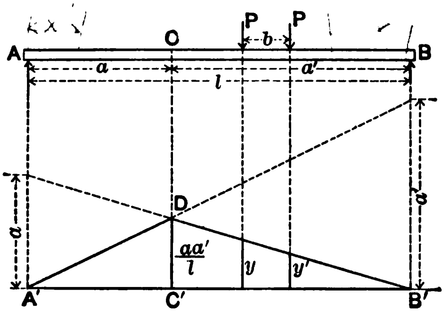


FIG. 10.

123. *Position of Loads for Maximum Moment.*—(*a*) *Uniform Load.*—The maximum moment due to a uniform load evidently occurs when the load extends from A to B, and is equal to the area $A'\ D\ B'$ multiplied by the load per unit length, or

$$M = \tfrac{1}{2}\, l \times \frac{a\,a'}{l} \times p = \tfrac{1}{2}\, p\, a\, a'. \quad . \quad . \quad . \quad . \quad (2)$$

(*b*) *Single Concentrated Load.*—Since the moment at C, due to any load, P, is $P\,y$, this moment is a maximum when the load is at C and is equal to $P\,a\,a'/l$.

(*c*) *Two Equal Loads.*—For two equal loads, P, a fixed distance, b, apart, the moment is evidently a maximum when one load is placed at the point and the other is on the longer segment of the beam, for then $y + y'$ is a maximum. This maximum moment is equal to

$$M = P\left(\frac{a\,a'}{l} + \frac{a\,a'}{l} \times \frac{a' - b}{a'}\right) = P\,\frac{a\,(2\,a' - b)}{l} \quad . \quad . \quad (3)$$

The above results for the three special loadings (*a*), (*b*), and (*c*) have been derived in Chapter IV, though in a different manner.

(*d*) *Any Number of Loads.*—Fig. 11 represents the beam loaded in any manner with a series of concentrated loads. So far as the bending

moment at C is concerned the loads between C and B produce the same effect as a single load G_2, equal to the sum of the several loads and applied at their centre of gravity; for the reaction at A remains the same and the moment $= R_1 \times a$. Likewise it may be shown that a single load G_1 will produce the same effect as the several loads between A and C. Hence the total bending moment due to the given system may be written

$$M = G_1 y_1 + G_2 y_2. \qquad (4)$$

To determine where the given system of loads should be placed to cause a maximum moment at C, consider what will be the effect upon the bending moment if the system is moved a small distance δx toward the left. Assume such movement to be made, no loads passing points A, C or B. The new value of the moment will be

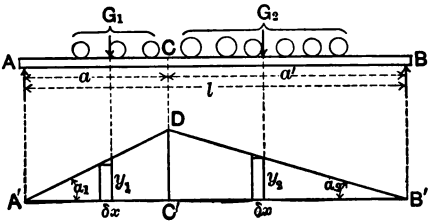


FIG. 11.

$$M + \delta M = G_1 (y_1 - \delta x \tan a_1) + G_2 (y_2 + \delta x \tan a_2), \qquad (5)$$

and the change in moment due to this movement is

$$\delta M = G_2 \, \delta x \tan a_2 - G_1 \, \delta x \tan a_1.$$

The rate of change per unit length is

$$\frac{\delta M}{\delta x} = G_2 \tan a_2 - G_1 \tan a_1 = \left(\frac{G_2}{a'} - \frac{G_1}{a}\right) \times C'D. \qquad (6)$$

For a maximum value of M, $\dfrac{\delta M}{\delta x}$ or $\dfrac{G_2}{a'} - \dfrac{G_1}{a}$ must change from positive to negative, that is, must pass through zero, as the loads are moved to the left. The values of G_2 and G_1 can change only when a load passes A, C or B. A load passing A decreases G_1 and a load passing B increases G_2, but a load passing C increases G_1 and decreases G_2; therefore it is only by the last method that $\dfrac{G_2}{a'} - \dfrac{G_1}{a}$ can be changed from positive to negative and the moment made a maximum. For a maximum moment, then, a load should be placed at C such that when

considered as part of G_2 the expression $\frac{G_2}{a'} - \frac{G_1}{a}$ is positive, and when considered as part of G_1 this expression is negative. Or, stated in another way, when G_1/a can be made equal to G_2/a' by considering the load which is at C as located partly on one side of C and partly on the other side. If a distributed load is passing C this condition can be definitely satisfied.

From the equality $\frac{G_2}{a'} = \frac{G_1}{a}$ we have, by composition, $\frac{G_2 + G_1}{l} = \frac{G_1}{a}$, or, if $G = G_1 + G_2$, we have the condition

$$\frac{G_1}{a} = \frac{G}{l}. \quad \quad (7)$$

This is the most convenient form of the criterion for maximum moment. Expressed in words it is that *the average load per unit length on the left of the point must be equal to the average load on the whole span.*

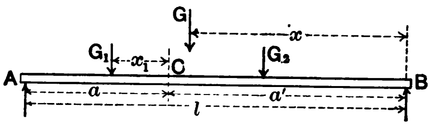


FIG. 12.

For a given set of wheel loads there are usually two or more positions which will satisfy this criterion. The moment for each position must be computed and the greatest value taken.

It is to be noted that if the criterion is satisfied when a load is at A or B, a minimum value will result instead of a maximum. Considering that the influence line to the left of A' and to the right of B' is a horizontal line, it may be stated in general terms that a maximum value requires a load to be placed at a point where the angle in the influence line is convex upwards and a minimum where it is concave upwards. This is a general principle applicable to all cases.

124. *Direct Algebraic Method of Deriving Criterion.*—Let x, Fig. 12, represent the distance of the centre of gravity of all the loads, G, from B, and x_1 the distance of G_1 from C. The moment at C is then equal to

$$M = G\frac{x}{l} \cdot a - G_1 x_1.$$

If the loads are advanced a distance δx towards the left, the *increase* in moment will be

$$\delta M = G\frac{a}{l}\,\delta x - G_1\,\delta x,$$

whence

$$\frac{\delta M}{\delta x} = G\frac{a}{l} - G_1.$$

This may be interpreted exactly as eq. (6) above. For a maximum we therefore have, as in eq. (7),

$$\frac{G_1}{a} = \frac{G}{l}.$$

125. *The Point of Greatest Moment in a Beam.*—Imagine a series of concentrated loads to be moving over a beam. The bending moment which exists under any given wheel will vary as the loads move, and will evidently be a maximum when that wheel is in the vicinity of the centre of the beam.

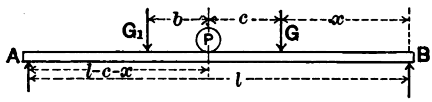


FIG. 13.

In Fig. 13 let P be any load in a given series. Let G represent the total load on the beam, and G_1 the load to the left of P. The distances c and b are independent of the position of the loads on the beam. The moment under load P is

$$M = G\frac{x}{l}(l - c - x) - G_1 b.$$

Differentiating and equating to zero, we find that for maximum moment, $2x = l - c$, or

$$x = l - c - x, \qquad (8)$$

that is, *the maximum moment under any given load occurs when that load and the centre of gravity of the series are equidistant from the ends (or centre) of the beam.*

Substituting the value of x from eq. (8) in the expression for M just above, we have a convenient expression for the value of the maximum moment,

$$M = G\frac{x^2}{l} - G_1 b. \qquad (9)$$

By means of eqs. (8) and (9) it is easy to find the maximum moment under any wheel; and if this is done for each of the wheels of the series,

the greatest of the resulting maxima will be the greatest possible moment in the beam. Practically it is necessary to find this maximum for only two or three wheels. To select the wheels to test, find, by the method explained in the preceding article, the wheel or wheels which should be placed at the *centre* to give a maximum moment at that point. The ones so found and those immediately adjacent are the ones to test for maximum moment.

126.—*Example of Moment Calculation.*—Let it be required to find the maximum centre moment in the beam of Fig. 8 for the loads there shown, also the point of greatest moment and the value of this moment.

Maximum Moment at C.—Try P_3 at C and test by eq. (7.) The total load $= G = 180$ and $\dfrac{G}{l} = \dfrac{180}{30} = 6.0$. The load to the left of C ($= G_1$), will range between a value of $P_1 + P_2$, when P_3 is considered on the right of C, to a value $P_1 + P_2 + P_3$ when P_3 is considered on the left; or $G_1 = 60$ to 100 and $\dfrac{G_1}{a} = \dfrac{60}{15}$ to $\dfrac{100}{15} = 4.0$ to 6.7. The former value being less than G/l, and the latter value greater, the selected position satisfies the criterion and the moment is a maximum. The reaction $= [20 \times 28 + 40\,(20 + 15 + 10 + 5)] \div 30 = 85.33$. The moment at C is $85.33 \times 15 - (20 \times 13 + 40 \times 5) = 820$. If P_4 be tried at C it will be found that P_1 will no longer be on the beam and the moment will be less than that already found.

The exact point of maximum moment will now be found in accordance with Art. 125. The centre of gravity of the five loads is found to be 9.22 ft. to the left of P_5 or .78 ft. to the right of P_3. In accordance with Art. 125, therefore, the maximum moment under P_3 will occur when that wheel is placed .39 ft. to the left of the centre, the centre of gravity being then an equal distance on the right. In this position the reaction $= R_1 = \dfrac{180 \times (15 - 0.39)}{30}$ and the bending moment under $P_3 = R_1 \times (15 - 0.39) - (20 \times 13 + 40 \times 5) = \dfrac{180 \times (14.61)^2}{30} - 460 = 824$. This is but slightly greater than the centre moment.

Further examples by the use of tabulations are given in Art. 145.

The moments can also be calculated readily by means of an equi-

librium polygon drawn for the given loads. In Fig. 14 is such a polygon constructed of indefinite length, with pole distance equal to 80. To find maximum M_c determine the proper position as explained above; P_3 is to be placed at C. Mark off the beam length, $A\,B$, so that C comes at P_3, and draw the closing line $a\,b$. The ordinate $c\,d \times H = M_c$. In the same way, by shifting the beam so that P_4 comes at C (shown as C'), the closing line $a'\,b'$ and ordinate $c'\,d' \times H$ gives the moment at C. The point of maximum moment and the value of this moment may be found by trial, or by first finding the position analytically and the moment graphically.

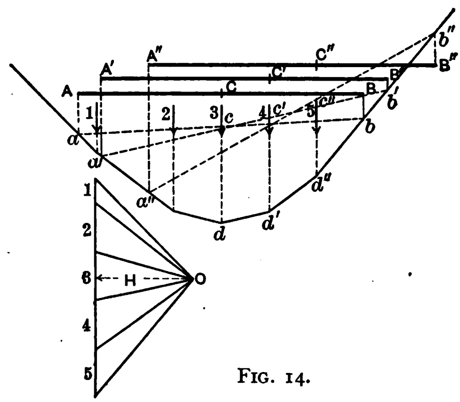


FIG. 14.

SECTION III.—TRUSSES WITH HORIZONTAL CHORDS AND SINGLE WEB SYSTEMS

127. Bending Moments at Joints of the Loaded Chord.—*Influence Line and Criterion for Maximum Moment.*—Consider joint C of the truss $A\,B$, Fig. 15. It has been shown (Art. 77) that the bending moment at a joint of the loaded chord is the same as in a solid beam under the same loading. Hence the influence line for this moment, and likewise the criterion for position of moving loads for a maximum, will be the same as in a solid beam. That is, in general, for maximum moment at C the condition is that $\frac{G_1}{a} = \frac{G}{l}$. When G_1/a is the lesser, then the moment will

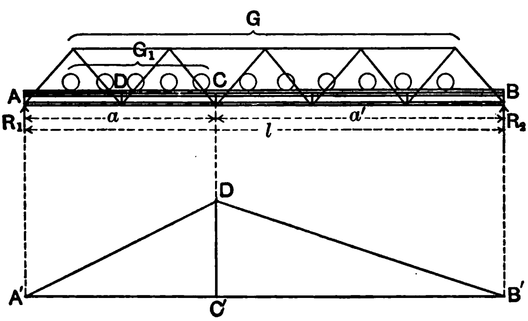


FIG. 15.

be increased by moving towards the left and when it is the greater then the moment is diminished by such movement. For a maximum, some wheel should be placed at C.

128. *Calculation of Moment.*—In calculating the value of the moment it will not be necessary to determine the actual joint loads, but the moment may be found from the wheel loads directly, as in a solid beam. This is shown by the influence line and can also be shown as follows:

The bending moment at $C = R_1 \times a -$ (moment of joint loads at A, D and C, about C). But in getting R_1, the moment of the group G about B is evidently the same as the moment of all the joint loads about

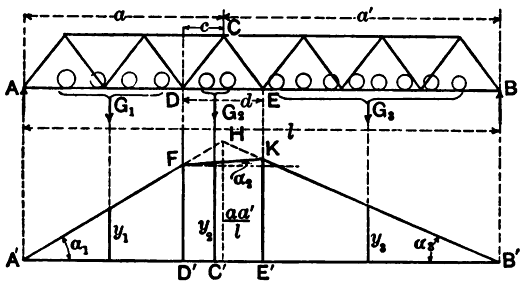


FIG. 16.

B, and the moment of the group G_1 about C is the same as the moment of the three joint loads at A, D, and C, about this point.

The bending moment at C can also be determined from the influence line by adding the products of the loads times the corresponding ordinates to the influence line when the loads are placed in the desired position. This method, however, possesses no advantage for such simple cases. The equilibrium polygon may also be used as in Art. 126. Further illustrations are given in Art. 142.

129. Bending Moments at Joints of the Unloaded Chord.—Where vertical web members are used, as in a Pratt Truss, the joints of the unloaded chord are located vertically above or below those of the loaded chord, and the moments are the same as at the joints of the loaded chord. Where, however, all web members are inclined, as in a Warren truss, the moments at joints of the unloaded chord will need to be separately considered.

Let $A\ B$, Fig. 16, be any truss with inclined web members, and C

any joint of the unloaded chord, not vertically above a joint of the loaded chord. Let c = horizontal distance from C to the next loaded chord joint towards A, and a = horizontal distance from C to A. The other notation is the same as that previously used.

130. *Influence Line.*—The effect upon the moment at C due to a load unity moving from B to E and from D to A is the same as for the moment at a point C in a beam $A\,B$; hence the influence line for that portion is $B'\,K$ and $F\,A'$, parts of the influence line $A\,H\,B'$, where $C'\,H = a\,a'/l$. Between E and D the load is carried by the stringer to these panel points. The influence line for this portion will then be the straight line $K\,F$. The values of the several angles are as follows:

$$\tan\alpha_1 = \frac{a'}{l};\ \tan\alpha_3 = \frac{a}{l};\ \tan\alpha_2 =$$

$$\frac{K\,E' - F\,D'}{d} = \frac{E'\,B'\tan\alpha_3 - A'\,D'\tan\alpha_1}{d} = \frac{c\,l - a\,d}{l\,d}. \quad (1)$$

131. *Position of Loads for Maximum Moment.*—To derive the criterion for maximum moment at C, let G_1, G_2, and G_3 represent the portions of the load that lie in the segments $A\,D$, $D\,E$, and $E\,B$, respectively. Let the loads advance a small distance $\delta\,x$ to the left. The increase in moment at C will be

$$\delta\,M = G_3\,\delta\,x\tan\alpha_3 - G_2\,\delta\,x\tan\alpha_2 - G_1\,\delta\,x\tan\alpha_1.$$

For a maximum,

$$\frac{\delta\,M}{\delta\,x} = 0 = G_3\tan\alpha_3 - G_2\tan\alpha_2 - G_1\tan\alpha_1. \quad (2)$$

Substituting from (1) we have

$$G_3\frac{a}{l} - G_2\frac{c\,l - a\,d}{l\,d} - G_1\frac{l - a}{l} = 0. \quad (3)$$

If $G = G_1 + G_2 + G_3$, we have, after reduction,

$$\frac{G}{l} - \frac{G_2\dfrac{c}{d} + G_1}{a} = 0, \quad (4)$$

or

$$\frac{G}{l} = \frac{G_2\dfrac{c}{d} + G_1}{a}. \quad (5)$$

For a maximum the left-hand member of eq. (4) must become zero by passing from positive to negative. This can only occur when some wheel passes E or D, as then only can $\dfrac{G_2 \frac{c}{d} + G_1}{a}$ be increased. Equation (5) is the criterion required.

132. *Calculation of Moment.*—In calculating the value of the moment at C the reaction and the negative moment of the group G_1 can be determined directly from the wheel loads, as in the case of the solid beam, but the effect of the loads G_2 must be separately considered. The partial joint load at D, caused by these loads, must be determined and the moment of this joint load taken about C.

The moment at C can also be found quite expeditiously by calculating the moments at D and E which occur for the same position of loads. Then, since the moment from D to E varies uniformly, we have, by interpolation,

$$M_C = M_D + (M_E - M_D) \frac{c}{d}. \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

133. **Maximum Floor-Beam or Joint Load.**—Let $A\,C$ and $C\,B$, Fig. 17, be two consecutive panels of the loaded chord, whose lengths are d_1 and d_2. Any load resting on the floor of the bridge is transferred by the longitudinal stringers to the cross-beams, or floor beams, and by them to the truss at the joints A, C, B, etc. It is desired to determine for any given loading the maximum load which may come upon the floor beam, or the maximum joint load on the truss.

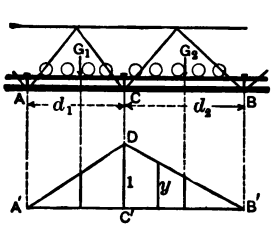


FIG. 17.

134. *Influence Line.*—A load of unity moving along the floor from B to C causes a joint load at B proportional to the distance of the load from C. The line $B'\,D$, with ordinate $C'\,D$ equal to unity, is then the influence line for floor-beam load at C when the load is between B and C. Likewise $D\,A'$ is the influence line for the portion $A\,C$.

An analysis similar to that in Art. 123 leads readily to the similar result, that for a maximum floor-beam load the condition is that

$\frac{Ga}{L} - G_1 = \frac{+}{0}$

$$\frac{G_1}{d_1} = \frac{G_2}{d_2}, \qquad (7)$$

or if $G = G_1 + G_2$,

$$\frac{G_1}{d_1} = \frac{G}{d_1 + d_2}. \qquad (8)$$

This is the same criterion as deduced in Art. 123 for bending moment at point C in a simple beam of length $d_1 + d_2$ (Fig. 18). This might have been concluded from the similarity of the influence lines of Fig. 17 and Fig. 10. In Fig. 18 the ordinate at $C = d_1 d_2/l$, while in Fig. 17 it is unity. However, in the determination of the criterion of Art. 123 this quantity is eliminated so that the result is the same. It may therefore be concluded that eq. (7) or (8) is, in general, the criterion for a maximum value for any function for which the influence line is of the form shown in these cases.

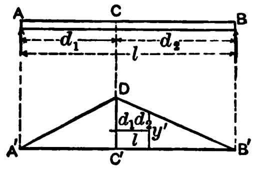


FIG. 18.

If $d_1 = d_2$, as is generally the case, then the criterion becomes $G_1 = G_2$, that is, the loads on the two panels must be equal. To satisfy the criterion a load is generally required at C.

135. *Calculation of Floor-Beam Load.*—The load at C is equal to the sum of the reactions at C for the longitudinal beams $C B$ and $C A$, due to the loads in the respective panels. It may be found by the usual moment equations; or it may be found as follows:

In Fig. 18 $A B$ is a beam of length $d_1 + d_2$ and $A' D B'$ is the influence line for bending moment at C. The ordinate $C' D = \frac{d_1 d_2}{d_1 + d_2}$. Comparing now the influence lines of Figs. 17 and 18 it is seen that for any two ordinates y' and y, similarly located, $y' : y = \frac{d_1 d_2}{d_1 + d_2} : 1$, or $y = y' \times \frac{d_1 + d_2}{d_1 d_2}$. Whence it follows that the floor-beam load at C (Fig. 17), due to any given loading is equal to the bending moment at C (Fig. 18), due to the same loading, multiplied by the factor $\frac{d_1 + d_2}{d_1 d_2}$. The maximum floor-beam load may therefore be found by first finding the maximum bending moment in a beam of length $d_1 + d_2$ at a distance d_1 from one end, and then multiplying this moment by $\frac{d_1 + d_2}{d_1 d_2}$. If $d_1 = d_2 = d$, the factor is $2/d$. This

method of calculation is often useful. (The student should prove the relation analytically.)

136. Shear in any Panel.—Let $A\ B$, Fig. 19, be any truss or a beam having a system of secondary members transferring the load to certain load points, E, C, etc. Let m' = number of panels to the right of C and m = number to the left.

137. *Influence Line.*—For a load unity placed between C and B the shear in the panel $E\,C$ is equal to R_1, and for a load between A and E the negative shear $= R_2$, as in a beam. The influence line from B to C and from A to E can then be constructed as in Fig. 6 (Art. 118)

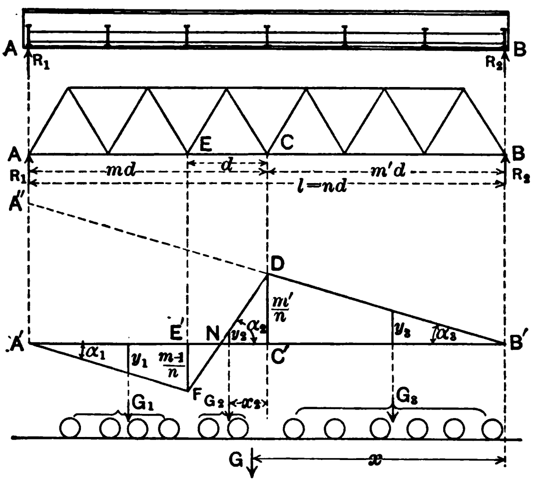


FIG. 19.

making $A'\,A''$ equal to 1 and drawing $A'\,F$ parallel to $B'\,A''$. The ordinate $C'\,D = m'/n$, and $E'\,F = (m-1)/n$. For the portion $C\,E$ the shear will vary uniformly as the unit load moves from C to E, as shown in Art. 112. The influence line will therefore be the straight line $D\,F$.

The slopes of the segments are given by

$$\tan\alpha_1 = \tan\alpha_3 = \frac{1}{l};\ \tan\alpha_2 = \frac{m' + m - 1}{n\,d} = \frac{n-1}{n\,d}. \quad (9)$$

138. *Position of Loads for Maximum Shear.*—For a maximum positive shear due to a single load the load should evidently be placed at C, and for maximum negative shear, at E.

A uniform load will cause a maximum positive shear when extending from B' to N, and the resulting shear will equal $p \times$ area $N\,D\,B'$. The distance $N\,C'$ is the distance x of Art. 82 (b). It is equal to $\frac{D\,C'}{\tan a_2} = \frac{m'\,d}{n - 1}$, as there given. The value of the shear is

$$V = p \times \tfrac{1}{2}\,\overline{C'\,D} \times \overline{N\,B'} = p \,.\, \tfrac{1}{2}\,\frac{m'}{n}\left(m'\,d + \frac{m'\,d}{n-1}\right)$$

$$= \tfrac{1}{2}\,p\,d\,\frac{m'^2}{n-1}, \quad . \quad . \quad . \quad . \quad (10)$$

the same as given in eq. (9), Art. 82 (b).

For concentrated loads, let G_1 represent the portion from A to E, G_2 that in the panel $E\,C$, G_3 that on the right of C, and G the total load. The total shear is represented by $V = G_3\,y_3 + G_2\,y_2 - G_1\,y_1$. Then, as in Art. 123, moving the loads to the left a distance $\delta\,x$, subtracting the original value, and dividing by $\delta\,x$, we have $\delta\,V/\delta\,x = G_3 \tan a_3 - G_2 \tan a_2 + G_1 \tan a_1$. Equating to zero and reducing, we have finally, for maximum shear, the condition,

$$G_2 = \frac{G_1 + G_2 + G_3}{n} = \frac{G}{n}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

That is, *for a maximum shear in any panel the load in the panel must be equal to the load on the truss divided by the number of panels.*

Algebraically this may readily be derived as follows:

Let x = distance from G to point B, and x_2 the distance from G_2 to point C. Then the shear $= V = G\,\frac{x}{l} - G_2\,\frac{x_2}{d} - G_1$. Differentiating this as in Art. 124, we have

$$\frac{\delta\,V}{\delta\,x} = \frac{G}{l} - \frac{G_2}{d},$$

and hence for maximum

$$\frac{G}{l} = \frac{G_2}{d}, \text{ or } \frac{G}{n} = G_2, \text{ as in eq. (11).}$$

In order to cause a maximum and not a minimum the value of $\delta\,V/\delta\,x$ must reach zero by passing from positive to negative as the loads are moved towards the left. This can only occur when a load passes C or A, but for the greatest shears there will be no load, as a

rule, on the portion $A\,E$; hence the maximum shear will occur when some wheel near the head of the train is placed at C, such as to satisfy the above criterion.

If we assume no loads on $A\,E$, the criterion can be stated at once by inspection of the portion of the influence line $N\,D\,B'$. As shown in Art. 134, whenever the influence line is of this form the criterion for maximum is $\dfrac{G_2}{N\,C'} = \dfrac{G_3}{C'\,B'}$, or $\dfrac{G_2}{N\,C'} = \dfrac{G_2 + G_3}{N\,B'}$. Substituting the values above given for $N\,C'$ and $N\,B'$, and noting that $G_2 + G_3 = G$, we have, as before, the criterion

$$G_2 \frac{n-1}{m'\,d} = G \frac{n-1}{n\,m'\,d'}, \text{ or } G_2 = \frac{G}{n}.$$

This principle is particularly useful where the influence lines are actually constructed, or the distance $N\,C'$ is determined graphically in any other manner.

Since a wheel passing C causes a large relative increase in G_2, it will be found that the maximum shears in several panels will occur with the same wheel at the panel point to the right. To find for what panels any particular wheel P placed at the panel point to the right will give a maximum shear, let G_2 represent the wheels in the panel other than the wheel P. Then if P at the right-hand panel point gives a maximum, the criterion requires that $G > n\,G_2$ and $< n\,(G_2 + P)$. Wheel P at the right-hand panel point will then give a maximum so long as the entire load upon the bridge lies between $n\,G_2$ and $n\,(G_2 + P)$.

139. *Calculation of Shears.*—For concentrated loads the shear is equal to the reaction at A minus the joint loads from A to E inclusive. The latter will equal the sum of the wheel loads from A to E, plus the partial joint load at E caused by the loads in the panel $E\,C$.

Section IV.—Methods of Calculation

A. Analytical Methods

140. **Wheel Load Tabulation.**—Where stresses are to be calculated for a specified system of concentrated loads it is convenient to prepare, first, a tabulation of these loads giving weights, spacing and various moment products. Fig. 20 is a convenient form of such tabulation

Spacing		8	5	5	5	9	5	6	5	8	8	5	5	5	9	5	6	5	10	10
No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Σ Dist.	0	8	13	18	23	32	37	43	48	56	64	69	74	79	88	93	99	104	114	124
P	12.5	25	25	25	25	16.25	16.25	16.25	16.25	12.5	25	25	25	25	16.25	16.25	16.25	16.25	25	25
Σ P	12.5	37.5	62.5	87.5	112.5	128.75	145	161.25	177.5	190	215	240	265	290	306.25	322.5	338.75	355	380	405
M	0	100	287.5	600	1,037.5	2,050	2,693.75	3,563.75	4,370	5,790	7,810	8,385	9,585	10,910	13,520	15,061.25	16,986.25	18,680	22,290	26,030
M′ (P_1 omitted)	0	0	125	375	750	1,650	2,231.25	3,026.25	3,770	5,090	6,510	7,522.5	8,660	9,922.5	12,420	13,888.75	15,748.75	17,380	20,805	24,480
ΣP′ (P_1 omitted)	0	25	50	75	100	116.25	132.50	148.75	165.	177.5	202.5	227.5	252.5	277.5	293.75	310.0	326.25	342.5	367.5	392.5

Spacing		10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
No.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Σ Dist.	134	144	154	164	174	184	194	204	214	224	234	244	254	264	274	284	294	304	314	324
P	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
Σ P	430	455	480	505	530	555	580	605	630	655	680	705	730	755	780	805	830	855	880	905
M	30,080	34,380	38,930	43,730	48,780	54,080	59,630	65,430	71,480	77,780	84,330	91,130	98,180	105,480	113,080	120,880	128,880	137,180	145,730	154,530
M′ (P_1 omitted)	28,405	32,580	37,005	41,680	46,605	51,780	57,205	62,880	68,805	74,980	81,405	88,080	95,005	102,180	109,605	117,280	125,205	133,380	141,805	150,480
ΣP′ (P_1 omitted)	417.5	442.5	467.5	492.5	517.5	542.5	567.5	592.5	617.5	642.5	667.5	692.5	717.5	742.5	767.5	792.5	817.5	842.5	867.5	892.5

FIG. 20.—Wheel-load Tabulation, Cooper's E-50 Loading.

It is calculated for a set of two consolidation engines followed by a uniform train load weighing 5,000 lbs. per foot, and is known as Cooper's E–50 loading. (See Art. 169 for discussion of various loadings.) All loads are given in thousands of pounds and all moments in thousands of foot-pounds. The loads given are one-half the total loads, being those on one rail.

The various longitudinal columns contain the following data:

1. Distance between successive wheels.
2. Number of wheel from the left end (placed on the load).
3. Summation of distances from wheel No. 1.
4. Weight of each wheel load, P.
5. Summation of loads from the left end $= \Sigma P$.
6. Summation of moments of all loads to the left of any given wheel taken about that wheel.
7. Same as 6, except that wheel No. 1 is omitted.
8. Summation of loads from the left with wheel No. 1 omitted $= \Sigma P'$.

The uniform load is considered as consisting of equal concentrated loads 10 feet apart. This assumption represents the actual train load quite as closely as does a uniform load and gives practically the same results, as will be seen in later examples.

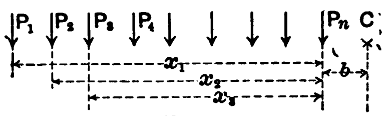


FIG. 21.

In calculating the moments of such tables, and in their application, the following principle is much used: In Fig. 21, let M_n represent the sum of the moments of all the loads from P_1 to P_n, taken about P_n as moment centre, and M_c represent a similar summation about any point C distance b from P_n. Then we have

$$M_n = \Sigma_1^n P\,x, \text{ and } M_c = \Sigma_1^n P\,(x + b) = \Sigma_1^n P\,x + \Sigma_1^n P\,b.$$

Hence

$$M_c = M_n + \Sigma_1^n P\,b. \qquad (1)$$

That is, the moment about any point beyond P_n is equal to M_n plus the product of the total weight times the distance from P_n to the point. If the value of the summation about the next load is desired then b is the distance from P_n to P_{n+1}.

141. Stresses in a Pratt Truss.—Let it be required to find the live-load stresses in the truss of Fig. 22, due to the loading given in Fig. 20.

142. *Maximum Moments.*—The maximum stresses in all chord members are determined from the maximum moments at b, c and d. The criterion for maximum moment at any point is, that the average load to the left of the moment centre must equal the average load on the entire bridge; or, as expressed in eq. (7), Art. 124, $\frac{G_1}{a} = \frac{G}{l}$. Taking a panel length as a unit, this becomes $\frac{G_1}{m} = \frac{G}{n}$, where m = number of panels to the left, and n = total number.

(*a*) Moment at b.—In general, for maximum moment, the bridge should be nearly covered with the live load, and some wheel must be placed at the moment centre. Try P_2 at b. The total load, G, which

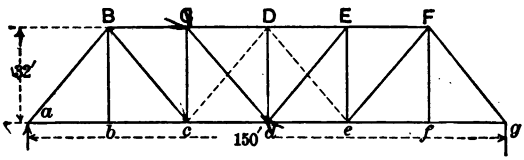


FIG. 22.

now comes upon the bridge may be determined as follows: The distance from P_1 to P_2 is 8.0 ft., and hence the distance from P_1 to the right end of the bridge = 8.0 + 125 = 133.0 ft. Looking along column 3 it is seen that P_{20} is the last wheel on the bridge and is 9 ft. to the left of point g. (A convenient way to get the location of the loads on the bridge is to lay off the wheels of the moment table of Fig. 20 to a true horizontal scale and mark off the panel points of the truss to the same scale on a separate strip of paper, which can be shifted along under the load diagram to any desired position.) The value of G is given in column 5 and is 405,000 pounds. Then $\frac{G}{n} = \frac{405}{6} =$ 67.5. The value of G_1 ranges from P_1 to $P_1 + P_2 = 12.5$ to 37.5; and $G_1/m = 12.5$ to 37.5. The value of G_1/m being less in both cases than G/n the moment will be increased by moving P_3 up to the joint. Try P_3 at b. P_1 is now 13 + 125 = 138 ft., from the right end and P_{21} is the last load on the bridge. $\frac{G}{6} = \frac{430}{6} = 71.7$ and $G_1/1 = 37.5$

to 62.5. The position does not give a maximum. Try P_4 at b. P_{21} is 9 ft. to the left of g, and $\frac{G}{6} = \frac{430}{6} = 71.1$. $G_1/1 = 62.5$ to 87.5. The former value being less than 71.1 and the latter value greater, a maximum moment results.

To calculate this moment get, first, the left reaction. This will be equal to the summation of moments about g, divided by 150 ft. The table gives 30,080 as the sum of the moments about P_{21}, hence as shown in eq. (1), Art. 140, $M_g = M_{21} + \Sigma_1^{21} P \times 9 = 30{,}080 + 430 \times 9 = 33{,}950$. Then $R_1 = 33{,}950/(6 \times 25)$. The bending moment at $b = R_1 \times 25$, minus the moment of joint load at a about b. The moment of this joint load is the same as the moment of the separate wheel loads, P_1, P_2 and P_3, about b, and is given in the tables under P_4. It is equal to 600. Then the moment at $b = R_1 \times 25 - 600 = 33{,}950/6 - 600 = 5{,}058$ thousand foot-pounds.

It is possible that P_5 will also give a maximum moment. With P_5 at b, $\frac{G}{6} = \frac{455}{6} = 76$ and $G_1/1 = 87.5$ to 112.5. This does not give a maximum. A continued movement of the loads will eventually cause a minimum moment, and later other maxima will result, with some wheel such as P_{13} or P_{14} at b.

Instead of assuming the train load to be made up of concentrations 10 ft. apart it may be treated as uniformly distributed. In this case the calculations will be somewhat modified. The head of the uniform train load is 109 ft. from P_1. Hence with P_4 at b the length of the uniform load on the bridge will equal $18 + 125 - 109 = 34$ ft., and will amount to $34 \times 2.5 = 85$ thousand pounds. The total load on the bridge is therefore $355 + 85 = 440$, instead of 430 previously used. The criterion for maximum is still satisfied. The moment of the loads about g is equal to the moment of the engine loads about g, plus the moment of the uniform load. The moment of the engine loads $= M_{18} + \Sigma_1^{18} P \times 39 = 18{,}680 + 355 \times 39 = 32{,}525$, and that of the uniform load $= 2.5 \times 34^2/2 = 1{,}445$. The total moment $= 32{,}525 + 1{,}445 = 33{,}970$, as compared to 33,950 by the former calculation. The moment at $b = 33{,}970/6 - 600 = 5{,}061$ instead of 5,058, a difference entirely negligible. The greatest difference arises when the centre of one of the assumed concentrations comes exactly at the end of the span.

In the above case it is one foot from this point. The difference in results by the two methods may therefore be neglected in general. If five-foot sections of the uniform load be taken, the maximum difference is reduced to one-fourth of that which results from the use of ten-foot sections.

(*b*) Moment at *c*.—The bridge should still be nearly covered but the load not quite so far advanced as for joint *b*, as the heavy drivers exert more influence than the other wheels.

Try P_6 at *c*. P_1 will be $32 + 100 = 132$ ft. from *g* and P_{20} will be 8 feet from *g*. $\frac{G}{6} = \frac{405}{6} = 67.5$. $\frac{G_1}{2} = \frac{112.5}{2}$ to $\frac{128.75}{2} = 54.2$ to 64.3. No maximum results.

Try P_7 at *c*. This will give a maximum and P_{21} will be 3 ft. to the left of *g*. The moment at $c = \frac{30{,}080 + 430 \times 3}{6} \times 2 - 2{,}694 = 7{,}763$.

P_8 at *c* gives no maximum.

(*c*) Moment at *d*.—P_{12} at *d* gives a maximum. P_{22} is then at *g* and $M_d = 34{,}380/6 \times 3 - 8{,}385 = 8{,}805$.

(*d*) Moment at *e*.—The moment at *e* for train headed towards the left is the same as moment at *c*, train headed towards the right. For spans as short as the one under consideration this moment may exceed that already found, but for long spans it will be less, as the moment centre will be located too far from the heavy drivers. In this case maximum values of M_e occur for P_{13} and P_{14} placed at *e*. The respective moments are 7,793 and 7,768. The maximum moment is therefore 7,793 and this is to be taken also as the maximum at *c*.

143. *Maximum Shears.*—Shear in panel *a b*. The criterion for maximum shear in this panel is the same as for maximum moment at *b*. Hence P_4 is to be placed at *b*. Then, as before, $R_1 = 33{,}950/150 = 226.3$. The joint load at *a* is equal to the moment of the wheels about P_4 divided by $25 = 600/25 = 24$. Hence shear $= 226.3 - 24 = 202.3$ thousand pounds.

Shear in panel *b c*. Try P_4 at *c*. For maximum shear $\frac{G}{l} = \frac{G_1}{d}$ or $G/6 = G_1$. In this case $G_1 = 62.5$ to 87.5 and $G/6 = 380/6 = 63.3$. Hence a maximum results. We also find that with P_3 at *c*

maximum is caused. For P_4 at c, $R_1 = \frac{22,230 + 380 \times 4}{150} = 158.3$. Joint load at $b = 600/25 = 24$. Shear $= 158.3 - 24.0 = 134.3$ For P_3 at c, $R_1 = \frac{18,680 + 355 \times 9}{150} = 145.8$. Joint load at $b = 287.5/25 = 11.5$ and shear $= 145.8 - 11.5 = 134.3$, the same as for P_4.

Shear in panel $c\,d$. P_3 at d gives the correct position and the shear is found to be 78.6.

Shear in panel $d\,e$. P_2 at e gives maximum positive shear, which is found to be equal to 37.1.

Shear in panel $e\,f$. P_2 at f is the correct position and the maximum value is 10.5.

144. *Floor-beam Load.*—For maximum floor-beam load the criterion is the same as for moment at the centre of a 50-ft. beam. P_4 placed at the centre of such a span gives values of G_1 equal to 62.5 to 87.5, and $G/2$ equal 72.5 to 80.6. With P_4 to the right of the centre, $G/2 - G_1 = 72.5 - 62.5 = +$, and with P_4 to the left of the centre, $G/2 - G_1 = 80.6 - 87.5 = -$. Hence this position is correct. The moment at the beam centre is 1,181.8. The floor beam load is now obtained from this moment as explained in Art. 135. It is equal to $1,181.8 \times 2/25 = 94.54$. This is also the stress in member $B\,b$.

The floor-beam load may also be found readily by the use of eq. (2), of Art. 148, for the calculation of panel concentrations. Counting the joint load desired as No. 2 of three successive joints, the value of n in this equation will be 2 and we have

$$W_2 = \frac{M_3 - 2\,M_2 + M_1}{d}.$$

With P_4 at No. 2, the values of the moment summations are,

$$M_3 = 3,563.75, \quad M_2 = 600 \quad \text{and} \quad M_1 = 0.$$

Whence

$$W_2 = \frac{3,563.75 - 1,200}{25} = 94.54$$

as before.

The load of 94,540 lbs. here found is the load on one rail, or on one truss of a single track bridge. It will be the end reaction of the floor beam, from which its moments and shears may be found.

145. *Stringer Moment and Shear.*—The maximum moment in a stringer is the maximum in a beam of 25-ft. span. By the same methods as used in Art. 126, the maximum moment is found to occur under P_3 when that wheel is placed 1.25 ft. to the left of the centre (the centre of gravity of the group being 1.25 ft. to the right). The value of the moment is 381.3. The moment at the *centre* with P_3, placed at the centre, is 375.0. The maximum end shear occurs with P_2 placed at the end. The shear = 70.65.

146. Moments in a Warren Truss.—In the Warren truss, Fig. 23, the moments at the joints of the unloaded chord, as C, are determined by the use of the criterion of eq. (5), Art. 131. In this case $c/d = \frac{1}{2}$, and the criterion is therefore $\dfrac{G_1 + \frac{1}{2} G_2}{a} = \dfrac{G}{l}$.

Suppose the moment at C is required. The distance $a = 50$ feet.

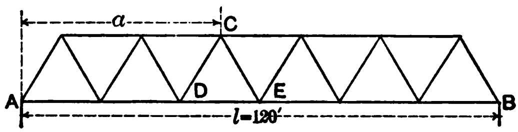


FIG. 23.

A load must be placed at D or E. Try P_6 at D. Considering P_6 to be at the right of D, then $G_1 = 112.5$, $G_2 = 177.5 - 112.5 = 65$, and $G = 355$. Hence $(G_1 + \frac{1}{2} G_2)/a = 2.90$. Considering P_6 on the left of D, $G_1 = 128.75$, $G_2 = 177.5 - 128.75 = 48.75$, and $(G_1 + \frac{1}{2} G_2)/a = 3.06$. $G/l = 355/120 = 2.96$. Hence this position gives a maximum moment. Try P_{10} at E. With P_{10} at E, the two values of $(G_1 + \frac{1}{2} G_2)/a$ are 3.07 and 3.19, and $G/l = 3.16$, hence a maximum. P_5 and P_7 at D, and P_9 and P_{11} at E were tried but did not answer the criterion.

In calculating the moment we may get the moment at C directly, or get the moments at D and E and interpolate. The latter method is the more expeditious. When P_6 is at D, P_{18} is 8 ft. to the left of the end, and $R_1 = \dfrac{18{,}680 + 8 \times 355}{120} = \dfrac{21{,}520}{120}$. Then $M_D = \dfrac{21{,}520}{120} \times 40 - 2{,}050 = 5{,}123.34$. With reference to joint E,

P_9 is 4 feet to the left. Then $M_E = \frac{21,520}{120} \times 60 - (4,370 + 177.5 \times 4)$ $= 5,680$, and $M_C = \frac{5,123.34 + 5,680}{2} = 5,401.67$. In the same way, we find that with P_{10} at E, $M_C = 5,401.67$. The maximum moment for both positions is, therefore, the same.

147. Pratt Truss with an Odd Number of Panels.—Where the number of panels is odd there is some difficulty in getting the exact maximum moments for the lower chord members of the centre panel. Fig. 24 shows a 7-panel Pratt truss. Assume *D e* and *d E* to be tension diagonals, so that in all cases the shear in the panel is carried by the member which is stressed in tension. When the shear is positive, *D e* will act and when negative *d E* will act. Suppose the loads so placed that the moment at *d* is a maximum. Under this loading the moment

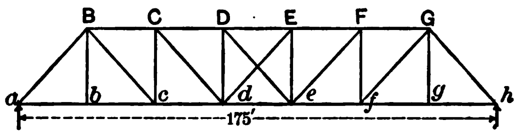


FIG. 24.

at *e* will be less than at *d*. The shear in panel *d e* will be negative (shear = differential of moment), and the diagonal *d E* will therefore be in action. The moment centre for *D E* is at *d* and its maximum stress is the same as in *C D*. For the member *d e*, however, the moment centre is at *E* when *d E* is in action and at *D* when *D e* is in action. Hence for the assumed position of loads the stress in *d e* will equal $M_E \div h$. But the given position of loads is not that which causes a maximum value of M_E and if the loads are shifted to produce maximum M_E then *De* comes into action and *D* becomes the moment centre of *d e*. Tracing the change in stress in *d e* as the loads are shifted from a position for maximum M_D to maximum M_E, it is to be noted that the stress in *d e* will increase slowly as long as *d E* is acting but as soon as *D e* begins to act it will decrease. The maximum stress in *d e* therefore occurs for such position of the loads as will cause equal moments at *d* and *e*, or zero shear in panel *d e*. This position can readily be found by trial, using a position intermediate between those for maximum moments at *d* and *e*.

It will be found that the maximum value of the stress thus determined is but little less than the maximum in $D\,E$. Generally it is assumed to be the same.

EXAMPLE.—Let the panel length of Fig. 24 be 25 ft., and use the loading of Art. 140. P_{12} at d gives maximum M_d whose value is 11,439; P_{14} at e gives maximum M_e whose value is 11,336. The latter position is 15 ft. farther to the right than the former. Try, therefore, a position with P_{14} placed 5 ft. to the left of e. For this position we find $M_d = 11{,}353$ and $M_e = 11{,}257$. The loads are evidently too far towards the left. Placing them with P_{14} 2 ft. to the left of e gives values of $M_d = 11{,}319$ and $M_e = 11{,}322$, which values are practically equal.

In this case the result could have been reached more quickly by first calculating M_d for P_{14} at e. This value = 11,249, which is but little less than the value M_e = 11,336, showing that the loads should not be moved far from this position.

It is to be noted that the value of 11,320 here found for maximum moment for member $d\,e$ is but one per cent. less than maximum M_d for member $C\,D$.

For such problems as this the graphical method by influence lines is convenient. In this case the shear influence line for the centre panel can be used and the loads placed by trial so that the shear ($= \Sigma\, P\, y$) is zero. The moment can then be found for this position.

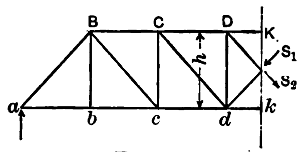


FIG. 25.

If the centre diagonals are members capable of resisting compression as well as tension (or are tension members adjusted with considerable initial stress), then the shear may be assumed equally divided between the two members. To find the stress in $D\,E$ (and $d\,e$ in this case), pass a section through the centre of the panel (Fig. 25), and take moments about k or K. The stresses S_1 and S_2 in the diagonals are equal and hence their moments about these points cancel. The stresses in $D\,E$ and $d\,e$, therefore, are equal to $M_k \div h$. The position for maximum M_k is found as in Art. 131, and is given in eq. (5). The moment is the same as at the centre joint of a 7-panel Warren truss. In the above example it is found to be 11,300. Compare this with the value of 11,320 previously found.

148. Computation of Panel Concentrations.—It is frequently required to compute all the joint loads or panel concentrations in a truss for some single position of the loading; as, for example, with wheel 3 at the first panel point from the left end. A convenient formula is readily derived as follows:

In Fig. 26, let W_0, W_1, . . . W_n, W_{n+1}, etc., be the panel concentrations; M_0, M_1, . . . M_n, M_{n+1}, etc., the summation of the moments of the loads to the left of each panel point about that point; b, the distance from panel point n to any other panel point; and d, the panel length. Suppose W_n is required. By equating the moments of the loads about n with the moments of the panel reactions we have

$$M_n = \Sigma_0^{n-1} W b, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

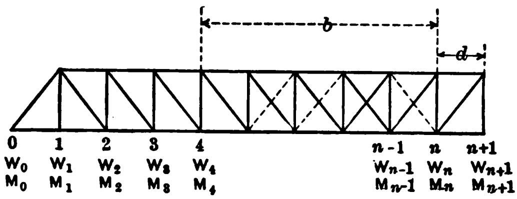


FIG. 26.

Similarly with centre at $n + 1$,

$$M_{n+1} = \Sigma_0^{n} W (b + d) = \Sigma_0^{n-1} W b + d\, \Sigma_0^{n-1} W + W_n d, \quad . \quad (b)$$

and with centre at $n - 1$,

$$M_{n-1} = \Sigma_0^{n-1} W (b - d) = \Sigma_0^{n-1} W b - d\, \Sigma_0^{n-1} W, \quad . \quad (c)$$

Substituting from (a) and (c) in (b) we readily get

$$W_n = \frac{M_{n+1} - 2 M_n + M_{n-1}}{d}. \quad . \quad . \quad . \quad . \quad (2)$$

The values of the M's can be taken from the diagram, or may be computed with the aid of the moment table, Fig. 20.

B. Graphical Methods

149. Graphical methods may be employed in various ways to assist in the analysis of stresses for concentrated loads. Graphics may be applied to the determination of the position of loads for maximum effect or they may be used for the determination of the moments and shears themselves. Two general methods will be considered: (1) the load line for finding positions, with the moment diagram or equilibrium polygon for finding moments and shears; and (2) the influence line for getting both positions of loads and values of functions. These methods are often combined to advantage.

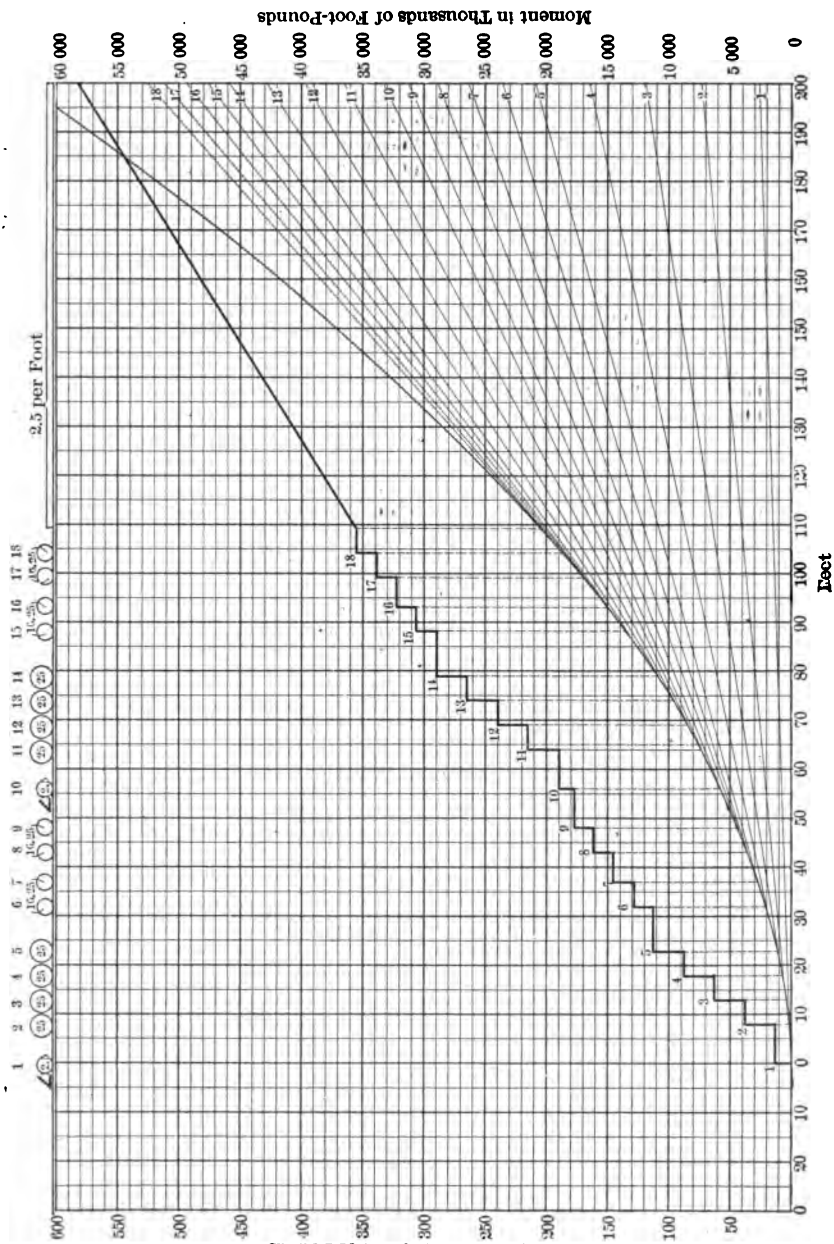


FIG. 27.—Load-Line and Moment Diagram, Cooper's E — 50 Loading.

150. Load-Line and Moment Diagram.—While these two diagrams are separate and distinct diagrams they are conveniently drawn on a single sheet. Fig. 27 shows such a set of diagrams constructed for Cooper's *E*–50 loading. They are drawn as follows: The wheel-diagram is first laid off along the top as shown. The heavy stepped line, or "load-line," is then constructed by measuring off ordinates from the zero line, 0 – 0, equal to the sum of the loads on the left up to the point in question. Under the wheels the load-line consists of a series of steps each step being equal to the weight of the wheel above. Under the uniform load the line may be drawn as a straight line, having a slope of 2,500 lbs. per foot.

The moment lines, numbered 1, 2, 3, 4, . . . 18, at the right edge, are constructed by beginning at the horizontal reference line at some point near the right end of the sheet and laying off successively on a vertical line the moment of each wheel about that point, beginning with wheel 1. Then the line 1 is drawn from the reference line under wheel 1 to the first point thus found; the line 2 through the intersection of line 1 with the vertical under wheel 2, and the second point so found; then 3 through the intersection of 2 with the vertical under wheel 3, and the third point; etc. A scale of one one-hundredth of the scale for loads will be found convenient. The ordinate at any point, from the reference line to any one of these moment lines, is thus equal to the sum of the moments about the point, of the loads up to and including the load corresponding to the moment line taken. Also, the ordinate at any point between any two lines, as between 2 and 8, is equal to the sum of the moments of wheels 3 to 8, inclusive, about that point. The broken line *A B*, formed of segments of moment lines, is evidently an equilibrium polygon for the given loads. The portion *B C* above *B* is a parabolic curve but may be constructed as a polygon by using concentrated loads as in the moment table of Art. 140.

151. Moments at Joints of the Loaded Chord.—Before using the diagram, the panel points of the truss, or the points along the beam where the moments are desired, are first marked off on a slip of paper to the same scale as the diagram. Suppose 1–2–3–4 . . . , Fig. 28, represent a portion of a load-line, and *A B* a truss, laid off on a separate slip of paper. Let it be required to find the position of loads for a maximum moment at *C*. The average load on the left, $= G_1/A\,C$,

must equal the average load on the bridge, $= G/A\,B$. Try wheel 5 at C by placing C under this wheel. The ordinate $F\,B' = G$, and $G/A\,B$ is the slope of the line $A'\,F$. The ordinate $I\,C'$ or $K\,C' = G_1$, and $G_1/A\,C$ is equal to the slope of $A'\,I$ or $A'K$. The slope of $A'F$ is seen to be greater than that of $A'\,K$ and less than that of $A'\,I$; therefore this position gives a maximum moment at C. Hence, in general, if the line whose slope is equal to G/l cuts the load or step which is above the point, the position is one giving a maximum. This line is conveniently indicated by means of a thread.

If there are no loads off the bridge to the left, the line $A'\,F$ starts from the zero line at the end of the bridge at A'; but if there are loads

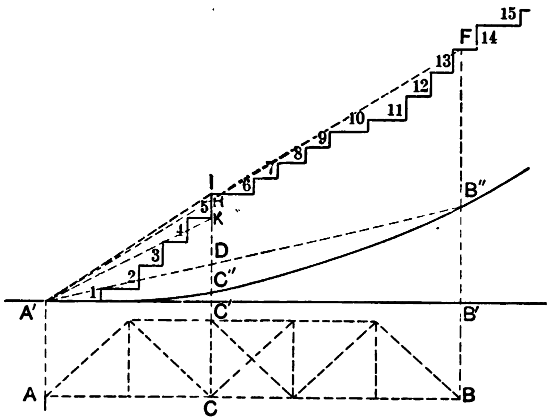


FIG. 28.

to the left, it starts in the load-line vertically over the end of the bridge. The right end is the point in the load-line vertically over the right end of the bridge. In the above case, if $A'\,F$ should pass above $A'\,I$, the loads must be moved to the left; and if it should pass below $A'\,K$, then the loads must be moved to the right.

The moment itself is easily found on the moment diagram by reading off the ordinate $B'\,B''$, multiplying this by $AC\;/A\,B$, and subtracting the moment of the loads to the left about C, which is given by the ordinate $C'\,C''$.

The moment at C is also equal to the ordinate $C''\,D$ from the closing line $A'\,B''$ to the equilibrium polygon formed by the segments

of the moment lines. The extremities of this closing line lie in verticals through the ends of the bridge.

In finding the greatest possible moment in a girder, the centre of gravity of any number of loads is readily found by producing the two segments of the equilibrium polygon including these loads, to their intersection.

152. Moments at Joints of the Unloaded Chord.—Fig. 29 shows a load-line and the outline of a truss *A B*. Try wheel 3 at *D* for a maximum moment at *C*. The slope of the line *A′ F* is equal to the left-hand member of eq. (5), Art. 131. G_1 varies between the values *D′ M* and *D′ M′*, or *C′ K* and *C′ K′*, according as wheel 3 is considered in or out

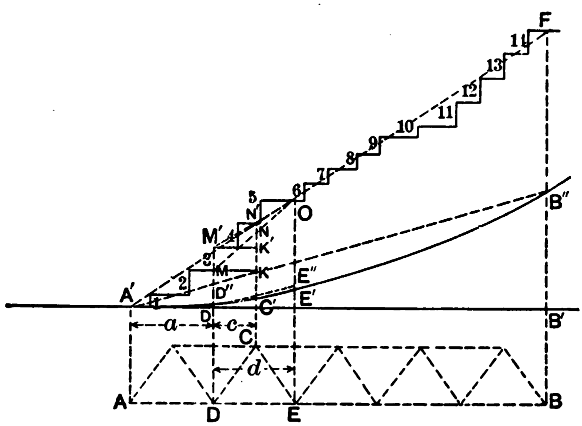


FIG. 29.

of the panel *D E*. The value of G_2 varies correspondingly between the sum of the loads 5, 4 and 3, and the sum of 5 and 4 only. The corresponding values of $G_2\, c/d$ are found by drawing *M O* and *M′.O*. They are *K N* and *K′ N′*. The slopes of the lines *A′ N* and *A′ N′*, if drawn, would therefore be the two values of the second member of eq. (5). Hence when *A′ F* cuts *N N′*, this position of the loads gives a maximum moment at *C*. None of the lines need be drawn, the points *N* and *N′* being lightly marked and a thread, *A′ F*, stretched from *A′* to the intersection of the vertical at *B* with the load-line. With a load at *E*, two points similar to *N* and *N′* are obtained by lines drawn from the intersection of the vertical through *D* with the load-line, to the upper and lower limits of the load at *E*.

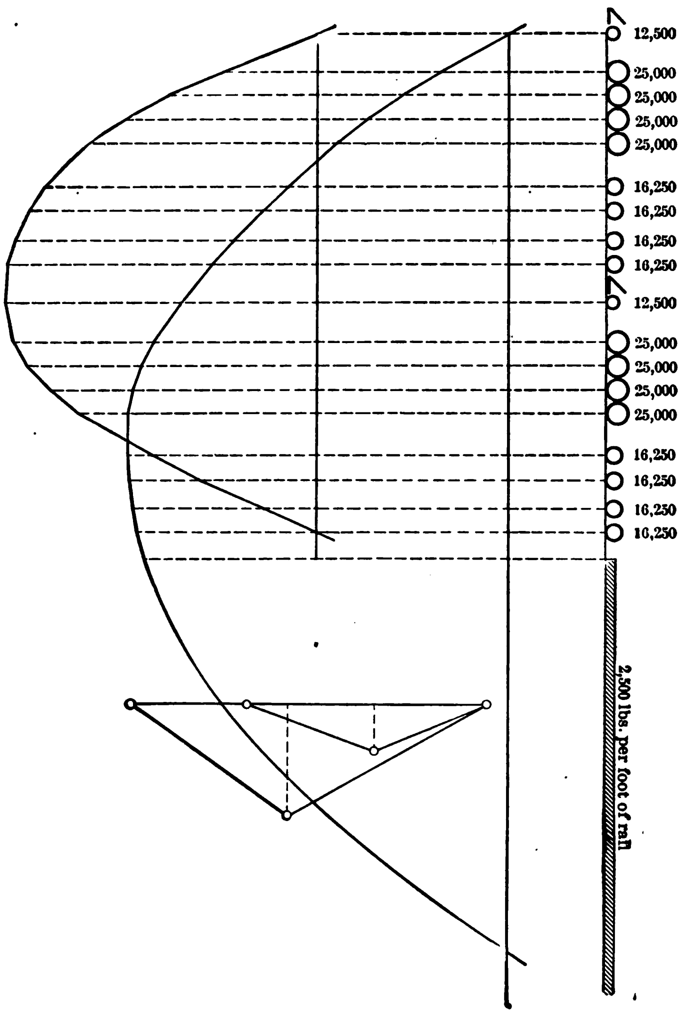

12,500
25,000
25,000
25,000
25,000
16,250
16,250
16,250
16,250
12,500
25,000
25,000
25,000
25,000
16,250
16,250
16,250
16,250
2,500 lbs. per foot of rail

FIG. 30

The position of the loads having been found, the bending moment at C is most readily found by proportion from the moments at D and E. Thus if M_D and M_E are the bending moments at D and E, respectively, found as above described, then the moment

$$M_C = M_D + (M_E - M_D)\frac{c}{d}.$$

Graphically, the moment at C is equal to the ordinate intercepted between the closing line $A' B''$ and the closing line $D'' E''$.

153. Use of the Equilibrium Polygon.—If it is desired to determine moments by measuring ordinates to an equilibrium polygon, a better form for such a polygon is shown in Fig. 30, the pole distance being selected so as to give ordinates of considerable length. To accommodate the work to spans of various lengths two or three polygons may be drawn of suitable portions of the train load, and with poles distances selected accordingly. Fig. 30 shows two such polygons, the shorter being for the two engines alone, and suited to spans up to about 100 ft., and the longer being drawn for 200 feet of train, and serving for spans from 100 to 200 feet in length.

154. Shears and Web Stresses.—(*a*) *Shear in Beams or Girders.*—It has been shown in Art. 119 and illustrated in Art. 120, that wheel 1 will cause a maximum shear at all points to the right of the point where $\frac{G'}{l} = \frac{P_1}{b}$, and that wheel 2 will cause a maximum shear to the left of the point where $\frac{G}{l} = \frac{P_1}{b}$; where G and G' are the total loads on the girder with wheels 1 and 2, respectively, at the point; b = distance between wheels 1 and 2; P_1 = weight of wheel 1; and l = length of girder.

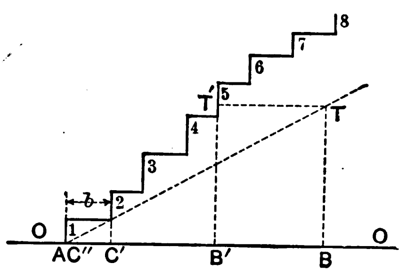


FIG. 31.

Let A 1-2 . . . 8, Fig. 31, be a load-line. Draw the line $A\,T$, having a slope of P_1/b. Lay off the length of the girder $A\,B$ on the line o–o, and at B draw the vertical $B\,T$. This vertical ordinate is that load which, divided

by $l, = P_1/b$, and therefore is the load G or G' above mentioned. In order that this may be the total load on the girder, wheel 5 must be at the right end. Laying off the girder then from B' to the left, the portion $B'C'$ to the right of wheel 2 is the portion of the girder in which the maximum shear is given by wheel 1, and the portion of the girder to the left of C'' has its maximum shears for wheel 2. Between C' and C'' both positions should be tested. The line $A\,T$ may be permanently drawn on the diagram; no other lines need be drawn.

(*b*) *Shear in a Truss.*—In Fig. 32, 1–2 . . . 14, is a load-line and $A\,B$ a truss, outlined on a separate strip of paper. Consider the maxi-

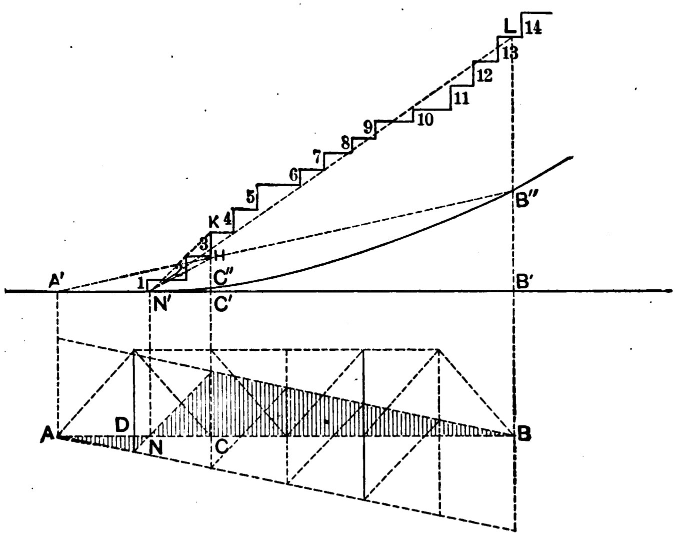


FIG. 32.

mum positive shear in panel $D\,C$. Construct the influence line for shear as in Art. 137, using the lower chord as a base line. The criterion for maximum shear is that $\dfrac{G_2}{N\,C} = \dfrac{G}{N\,B}$. Try P_3 at C, as shown in the figure. The slope of $N'L$ is $G/N\,B$ and the slopes of $N'H$ and $N'K$ are the two values of $G_2/N\,C$. Hence if $N'L$ cuts $K\,H$, the position gives a maximum.

The influence lines for all the panels can be drawn on the same outline as shown.

The values of the shears are readily obtained from the diagram. Reaction = ordinate $B'B'' \div l$, and joint load at D = ordinate $C'C'' \div d$. Then $V = R_1 -$ (joint load at D).

155. General Use of the Influence Line.—Let Fig. 33 represent any influence line consisting of several segments. Let α_1, α_2, etc., be the inclinations of the several segments measured from the horizontal. Let G_1, G_2, etc., represent the total load resting upon the portions of the structure corresponding to the several segments.

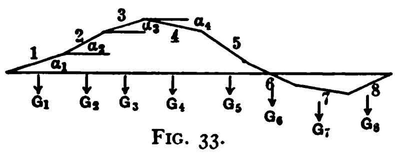


FIG. 33.

From the analysis of Art. 123, it is evident that the general form of the criterion for a maximum is, taking due account of sign,

$$\Sigma G \tan \alpha = 0, \qquad (3)$$

In cases such as those already considered, a simple criterion can be deduced from this general expression. In the more complex cases, as in some of the subsequent examples, this is not practicable, and the position giving a maximum must be determined by trial. In doing this, various positions may be tried, the actual value of the function worked out for each position and the results compared. It is easier in some cases, however, to calculate the value of the first member of eq. (3). If this is positive, then the loads should be moved toward the right, and if negative then toward the left. Graphically, this quantity may be obtained as follows: On a horizontal line $A\ B$, Fig. 34, lay

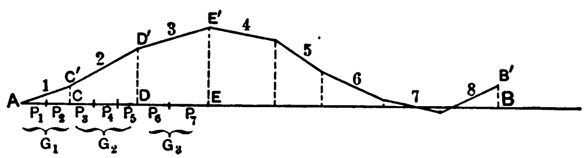


FIG. 34.

off to any convenient scale the weights of the given series of concentrated loads, P_1, P_2, etc. This may be done once for all. Then by means of a space diagram of wheels, applied to Fig. 33, the particular loads located under each segment of the influence line can be noted, thus determining G_1, G_2, etc., of Fig. 34. Then draw $A\ C'$ parallel to segment 1 of Fig. 33, $C'\ D'$ parallel to segment 2, etc. The closing ordinate $B\ B'$ will then equal $\Sigma G \tan \alpha$.

In cases such as here considered, the influence line furnishes the most convenient method of determining the values of the desired functions themselves. For this purpose it should be accurately drawn to a large scale. The value of the function for any given position of loads is then simply the sum of the products of the several loads times the corresponding ordinates to the influence line.

Section V.—Trusses with Inclined Chords and Single Web Systems

156. Chord Stresses.—The methods already described in Arts. 127–129, for determining maximum moments, apply also to any truss with single web system. The maximum chord stresses are determined from maximum moments.

157. Web Stresses in the Curved-Chord Pratt Truss.—In trusses with inclined chords the web stress is not dependent upon the shear alone, but is a function of chord stress as well. The method of Art. 136 is, therefore, not applicable to this case.

158. *Influence Lines.*—Consider the diagonal $E\,D$, Fig. 35. By

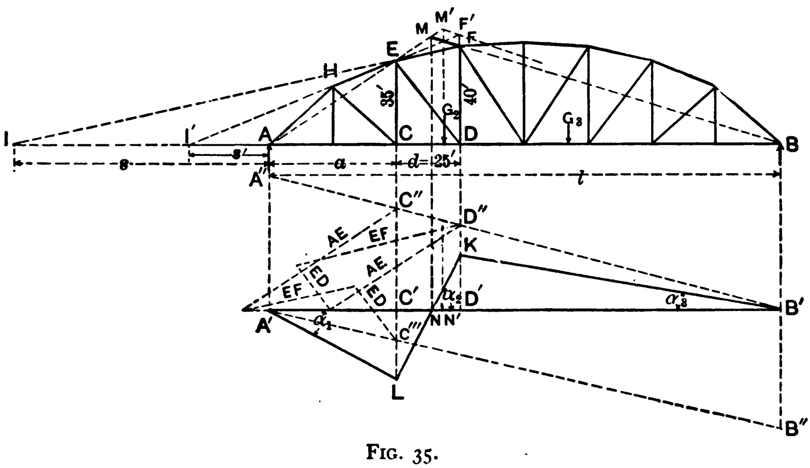


Fig. 35.

either of the methods explained in Art. 93, calculate the stress in $E\,D$ for a load unity at D and again at C, and lay off the ordinates $D'\,K$

and $C' L$ equal to these stresses. The line $A' L K B'$ is the influence line for $E D$.

The point N, where the influence line crosses the axis is conveniently determined by drawing $A E$ and $B F$ to an intersection M. The point N is vertically below M. It is the point where a load causes zero stress in $E D$. The correctness of the construction may be shown as follows: For a load placed at N and supports at A and B, the lines $A M$, $M B$, and $B A$ will represent to some scale the diagram of moments, or the equilibrium polygon. The ordinates will therefore also represent to some scale the bending moments at the joints of a truss. Hence we may write $M_E = EC$ and $M_D = DF$. The stress in $C D$, therefore, $= \dfrac{\text{Mom. } EC}{\text{Length } EC}$, and the hor. comp. in $E F = \dfrac{\text{Mom. } DF}{\text{Length } DF}$. These two quantities are equal, hence the stress in $E D$ for this loading is zero. In some methods of analysis the point N is all that is needed of the influence line.

The numerical calculation of $E D$ by shears from the influence line is very simple. For 1 lb. at D, V. comp. $E F = R_1 \times \frac{3d}{40} \times \frac{5}{d} = R_1 \times 3 \times 5/40$. V. comp. $E D =$ shear − V. comp. $E F = R_1 [1 - (3 \times 5)/40]$, etc. For unit load at C, V. comp. $E F = R_2 \times \frac{5d}{40} \times \frac{5}{d}$, and shear $= R_2$. V. comp. $E D =$ shear + V. comp. $E F$.

A graphical construction is sometimes convenient. For this purpose draw the reaction lines $A'' B'$ and $A' B''$. The ordinate $D' D'' = R_1$ for unit load at D. Then to get the stress in $E D$ use the method explained in Art. 50, Chapter II. Replace the truss between A and $E C$ by the triangle $A E C$ and draw the force polygon for joint A. It will be completed by drawing the line $\overline{A E}$ from D''. Then draw the diagram for joint E, the stress in $E C$ being zero. The lines $\overline{E F}$ and $\overline{E D}$ give the desired stress, $\overline{E D}$, to be laid off as $D' K$. For a unit load at C the reaction is $C' C''$. As before, $\overline{A E}$ gives the stress in $A E$. When we come to joint E the stress in the member $E C$ is unity, equal to the load at C, and given by the length $C'' C'''$. The diagram for this joint is therefore completed by drawing from C''' the lines $\overline{E D}$ and $\overline{E F}$, thus again determining the stress in $E D$. It is compressive and is laid off as $C' L$.

159. *Position of Loads for Maximum Web Stress.*—The maximum stress in $E\,D$ occurs when some of the wheels at the head of the train are in the panel $C\,D$, and, in exceptional cases only, when some of the loads are to the left of C. In the cases treated we will assume that there are no loads on the portion $A\,C$.

Let G_2 represent the total load in the panel $C\,D$; G_3, the load in $D\,B$; and G the total load on the bridge. Let the loads advance from any given position a distance δx towards the left. The stress in $E\,D$ will be increased by an amount $\delta M = G_3\,\delta x \tan\alpha_3 - G_2\,\delta x \tan\alpha_2$. For a maximum

$$\frac{\delta \mathrm{M}}{\delta x} = 0, \text{ or } G_3 \tan\alpha_3 = G_2 \tan\alpha_2. \quad . \quad . \quad . \quad (a)$$

From this we may write, as in Art. 123, the criterion for maximum,

$$\frac{G_3}{D'\,B'} = \frac{G_2}{N\,D'} \text{ or } \frac{G}{N\,B'} = \frac{G_2}{N\,D'}. \quad . \quad . \quad . \quad . \quad (1)$$

This is the most convenient form of the criterion when the point N is known. It is also the most convenient form to use with the load line, as shown in Art. 154 (*b*).

For analytical methods the value of $N\,D'$ may be desired. The point N is determined analytically by equating to zero the stress in $E\,D$ (or moment at I), for a load unity at N. We have from this, $\frac{N\,B'}{l}\cdot s - \frac{N\,D'}{d}(s+a) = 0$, whence $\frac{N\,D'}{N\,B'} = \frac{d\,s}{(s+a)\,l}$. Substituting this in eq. (1), we derive the form of criterion

$$\frac{G}{l} = \frac{G_2\left(1 + \frac{a}{s}\right)}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is quite similar to the criterion for shear of Art. 138, the term (a/s) taking care of the effect of inclined chords.

For maximum stress in the vertical $E\,C$, the same general position of loads is required, but they will need to be shifted slightly. The moment centre is now at I'. The same general criterion as given in eq. (1) will apply but the point N will be shifted to N'. Its position in this case is determined graphically as follows: Produce $H\,E$ to intersect $D\,F$ at F' and draw $A\,E$ and $B\,F'$ to intersect at M'. The

desired position of the point N' is now vertically below M'. In this construction we may consider the member EF replaced by EF', which gives a truss in which EC and ED are both zero when the stress in HE = stress in EF', hence the correctness of the construction.

For the vertical EC, the criterion of eq. (2) is changed by replacing s by s' of Fig. 35.

In numerical work the various values of s are conveniently expressed in panel lengths. For stresses in members to the right of the centre (counters and minimum stresses), the intersection, I, lies to the right of B, and s is negative and measured from A.

160. *Calculation of Stresses.*—(*a*) Analytical Method.—For uniform loads the maximum stress = area $NKB' \times p$. If $u = KD'$, = stress in ED for 1 lb. at D, then

$$\text{Maximum } ED = \tfrac{1}{2}u \times NB' = \tfrac{1}{2}u\left(\frac{l-a-d}{1-\dfrac{d}{\left(1+\dfrac{a}{s}\right)l}}\right) \quad . \quad . \quad (3)$$

For concentrated loads the calculation of stresses will be illustrated

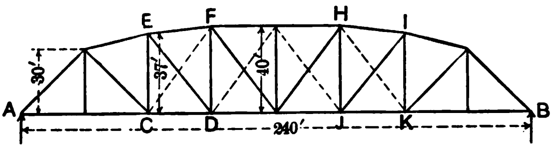


FIG. 36.

by the determination of the maximum stresses in members EC, ED and HK, Fig. 36. The loading of Fig. 20 will be used.

Member EC.—For this member the value of s, of eq. (2), expressed in panel lengths, is equal to $(30/7 - 1)\,d = 3.285\,d$. $a = 2\,d$. Hence

$1 + \dfrac{a}{s} = 1 + \dfrac{2}{3.285} = 1.61$. The criterion for maximum stress is, therefore, $G/8 = 1.61\,G_2$.

Try P_2 at D. $\dfrac{G}{8} = \dfrac{480}{8} = 60$. $G_2 = 12.5$ to 37.5 and $1.61\,G_2 =$ 20.1 to 60.3. The position gives a maximum. The value of R_1 is found to be 170.2 and the joint load at $C = 3.33$. The stress in EC is then found from the equation

$$EC = 170.2 \times \frac{3.285}{5.285} - 3.33 = 102.5 \text{ compression.}$$

Member ED.—The value of $s = (37/3 - 2)\,d = 10.33\,d$. $1 + a/s = 1.19$. Placing P_3 at D we have $G/8 = 60$ and $1.19\,G_1 = 44.6$

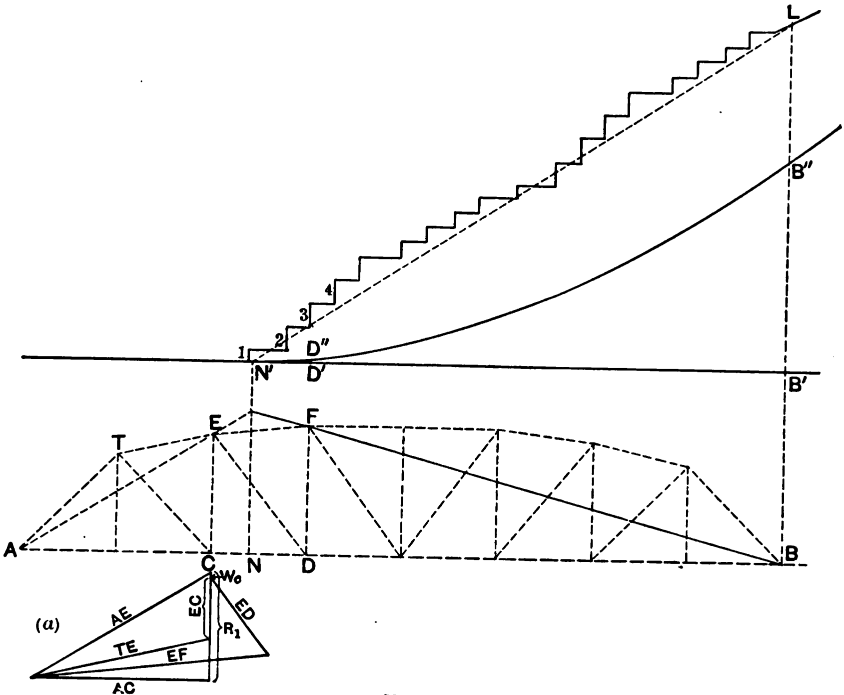


FIG. 37.

to 74.4. The position gives a maximum. Then $R_1 = 180.2$ and joint load at $C = 9.58$.

$$V. \text{ comp. } ED = \frac{180.2 \times 10.33\,d - 9.58 \times 12.33\,d}{13.33\,d} = 130.8.$$

$$\text{Stress} = 130.8 \times \frac{\sqrt{20^2 + 37^2}}{37} = 168.1 \text{ tension.}$$

Member HK.—The members HI and JK intersect at a distance $10.33\,d$ from B or $18.33\,d$ from A. Hence $s = -18.33\,d$. $a = 5d$, hence $1 + a/s = 1 - .273 = 0.727$.

Placing P_2 at K we have $G/8 = 26.9$ and $.727\,G_2 = 9.1$ to 27.3.

hence this is the correct position. Then $R_1 = 34.04$ and joint load at $J = 3.33$.

$$\text{V. comp. } H K = \frac{34.04 \times 18.33\, d - 3.33 \times 13.33\, d}{12.33\, d} = 47.0.$$

$$\text{Stress} = 47.0 \times \frac{\sqrt{30^2 + 40^2}}{40} = 58.75 \text{ tension.}$$

(*b*) Graphical Method.—Fig. 37 shows a load-line and moment polygon, and the truss $A B$ outlined on a separate strip of paper. Consider the member $E D$. Find the point N as before described. Try P_3 at D, as shown, and draw the line $N' L$. If this cuts load 3 the position gives a maximum, as in Art. 154. The reaction $R_1 = B' B'' \div l$, and joint load at $C = D' D'' \div d$. The work can be completed graphically by the force polygon shown in Fig. (*a*). R_1 is the reaction and W_c is the joint load at C.

For the vertical $E C$ the point N' is a little towards the right from N, but the same position is likely to give a maximum stress as for $E D$. After getting the value of R_1 and joint load at C, the force polygon is drawn as shown in Fig. (*a*). R_1 and W_c are laid off as before, and the stress $A E$ found. Its H. comp. will be the same as the H. comp. of the member $T E$ in the actual truss. Drawing $\overline{T E}$ determines its stress and the stress $\overline{E C} = R_1 - W_c -$ V. comp. $\overline{T E}$ as shown.

161. *Maximum Tension in a Vertical where Counters are Used.*—This problem, solved by trial in Art. 96, Chapter IV, can more readily be solved by the use of influence lines. Consider the vertical $E C$, Fig. 38. Its maximum tension will occur when the load moves on from the left and advances to such a position that the stress in both the diagonals, $C F$ and $E D$, will be zero. The stress in the vertical will then equal the difference of the V. comps. in $H E$ and $E F$.

Construct the influence line for stress in $C F$ or $E D$. In the figure the full graphical analysis for $E D$ is given. The dead-load stress in $E D$ will equal (area $N K B'$ − area $A' N L$) × dead load per foot, and will be tension. Suppose the live load to consist of a moving uniform load of p per unit length. Placing a live load on the portion $A' N$ will cause maximum compression in $E D$, and equal to area $A' L N \times p$. The net compression in $E D$ will equal the live-load compression minus the dead-load tension. To reduce this to zero the

live load must advance to a point T so that the area $N\,T\,T' \times p$ will equal this net compression. Expressed algebraically, if w = dead load per foot we must have $p\,(N\,T\,T') = p\,(A'\,L\,N) - w$ is $(N\,K\,B' - A'\,L\,N)$. This position then gives a maximum tension in $E\,C$.

To determine the stress in $E\,C$ construct its influence line $A_1\,B_1\,C_2$ by calculating the stress for unit load at C, noting that $E\,D$ is not in action. (The H. comp. of both $H\,E$ and $H\,F = R_1 \times 2\,d/h$.) The construction is also shown in the figure. Finally the desired live-load

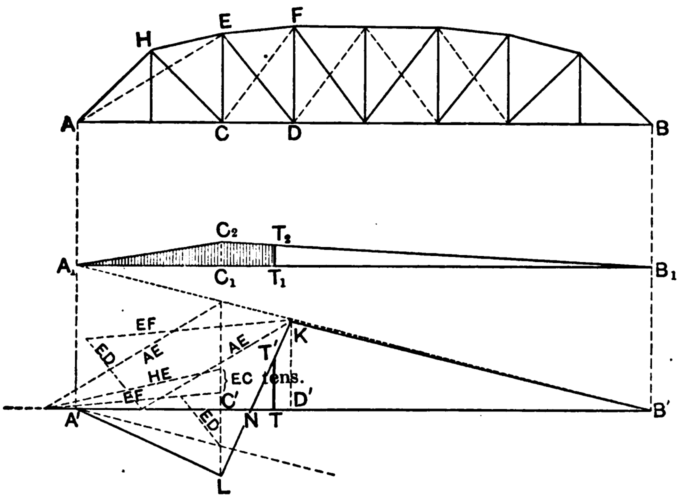


Fig. 38.

tension in $E\,C$ is the area $(A_1\,C_1\,T_1\,T_2C_2) \times p$. The dead-load stress is the joint load at E.

If concentrated loads are to be used, then instead of areas the summation of products, $\Sigma\,P\,y$, must be used, and the position for zero stress in $C\,F$ determined by trial. This position having been found, the stress in $C\,E$ is obtained from its influence line. The tensile stress in the vertical being of minor consequence it is sufficiently accurate to use an equivalent uniform load in this calculation in place of specified wheel loads.

162. General Treatment of Web Stresses in Trusses with Inclined Chords.—Let $A\,B$, Fig. 39, represent any single intersection truss.

Assume the lower joints to be the load points. Consider the member CF. The influence line will be of the same general form as in the case already considered. The point N is found by producing EF (the upper chord member cut by the section) to KL and drawing the lines KC and LD to intersect at M. The correctness of this construction will now be shown. The figure KLM is an equilibrium polygon for load P, with supports at A and B, and KL is the closing line. Likewise CMD is an equilibrium polygon for load P, with supports at C and D, and CD is the closing line. Hence the moment at F, due to P, $= h \times$ pole distance, and moment at $C = h' \times$ pole distance. The horizontal components of the stresses in CD and EF are equal

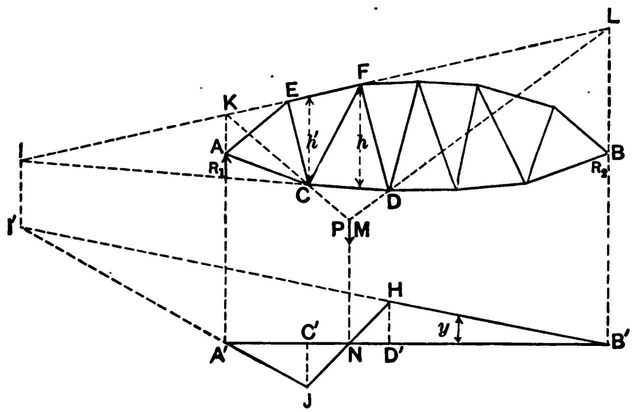


FIG. 39.

to the respective moments divided by the arms h and h'; they are therefore equal to each other under the assumed conditions and the stress in CF is zerc.

The ordinates to the influence line at D' and C' may be found, as in Art. 158, by placing unit loads at D and C and getting the stresses graphically. The triangle FKC may be substituted for the truss up to the member CE.

If a series of influence lines is to be drawn for various members, it is convenient to construct them by locating the point N and then determining the value of an ordinate y, for each diagram, by constructing a stress diagram of the entire truss for load unity at some one joint. The ordinate y being known, and the point N, the entire influence

line can be drawn. The stress diagram gives the value of y for all members.

An interesting relation between the influence line and the truss is the fact that the lines $B'H$ and $A'J$ intersect in the vertical through I. This is so because the ordinates to $B'I'$ represent the moment of the reaction R_1 about I, due to an advancing load, and the ordinates to $A'I'$ represent the moment of reaction minus the moment of the load about this point; when the load reaches I, if this could occur, these two quantities would be equal.

Using the point N, the criterion for maximum web stress is the same as in Art. 138, and the stresses may be found in the same way as there described.

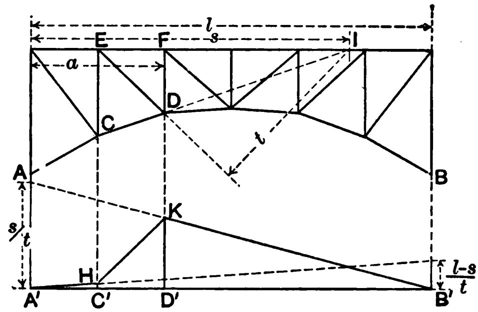


FIG. 40.

A truss of the form shown in Fig. 40, in which one chord is convex towards the other, presents conditions unlike those of the usual form. Consider the web member $E\,D$, so located that the two chords cut, intersect at a point I to the left of the reaction. For a load advancing from the right up to F the stress in $E\,D = R_1\,s/t$ and is tensile. It will increase uniformly until the point F is reached. The influence line for this position is $B'K$. The ordinate to the line at A' would be s/t. For a load advancing from A' to E, the stress will also be tensile and equal to $R_2\,(l - s)/t$, a quantity which becomes equal to $(l - s)/t$ at point B'. The entire influence line will therefore be $A'HKB'$, drawn as indicated. The structure should be fully loaded for maximum stress. If the chord segments intersect outside of the abutments then the case is the usual one already treated.

SECTION VI.—TRUSSES WITH SUBDIVIDED PANELS—TRUSSES WITH MULTIPLE WEB SYSTEMS

163. The Pettit Truss.—In the Baltimore and Pettit trusses the subdivision of the panel affects several of the members, causing a

considerable variation in form of influence lines from those for similar members in the Pratt truss. The Pettit truss with subties presents somewhat greater difficulties than any other form of these two types of

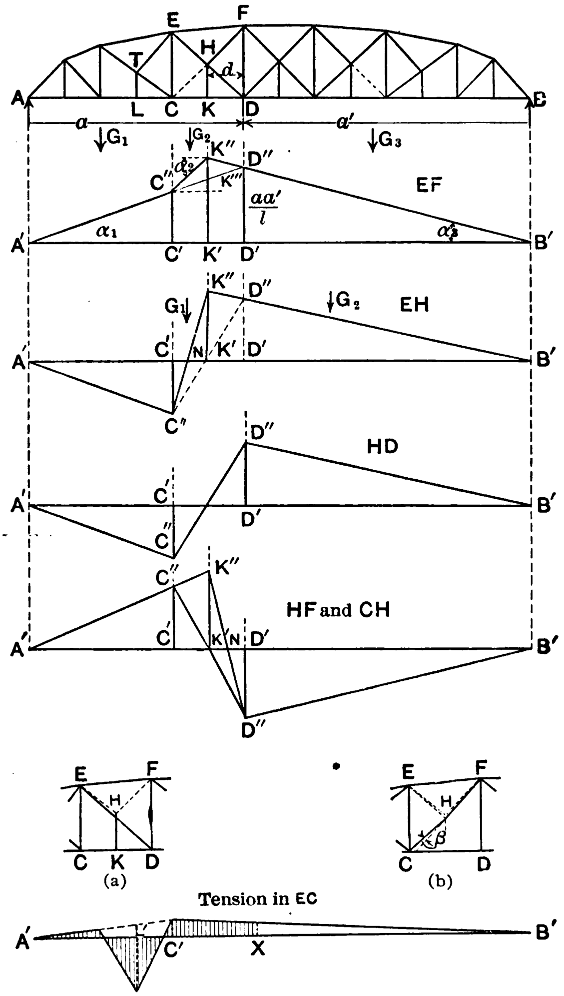


FIG. 41.

trusses, so that a complete analysis of this form will enable the student readily to analyze any of the others. The various members of the panel $C\ D$, Fig. 41, will be taken up in order.

The maximum stress in $H K$ is equal to a maximum joint load and the corresponding stress in $H F$ is readily found from $H K$. The piece $H F$ must also be considered as a counter; this is done later. The stress in $C D$ is obtained from the maximum moment at E. These members require no further consideration.

Member $E F$.—The centre of moments is at D, but note that the joint K is on the *right* of the section. The influence line for this moment will be the same as for moment at D, in a beam $A B$, except for the portion $C D$. The ordinate $D' D'' = aa'/l$.

As the load moves from D to K it is still on the *right* of the section and hence its effect will follow the law of the line $B' D''$ up to K''. The straight line, $K'' C''$, completes the influence line.

To determine the criterion for a maximum, let G_1, G_2 and G_3 represent the groups of wheels on $A C$, $C K$ and $K B$. For a maximum,

$$G_1 \tan \alpha_1 + G_2 \tan \alpha_2 = G_3 \tan \alpha_3, \quad . \quad . \quad . \quad . \quad (1)$$

The values of the tangents are

$$\tan \alpha_1 = \frac{a'}{l}; \ \tan \alpha_3 = \frac{a}{l}; \ \tan \alpha_2 = \frac{2\,d \tan \alpha_1 + d \tan \alpha_3}{d} = \frac{l + a'}{l} \quad (2)$$

Substituting in eq. (1), and reducing, the criterion becomes

$$\frac{G_1 + 2\,G_2}{a} = \frac{G}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To satisfy this condition a load must be placed at K.

In calculating the moment note that the influence line $A' D'' B'$ is the influence line for the moment at D in a beam $A B$, and the line $C'' K'' D''$, referred to $C'' D''$ as base, is the influence line for double the moment at K in a beam $C D$ (the ordinate $K'' K''' = d$). Hence the total moment is readily obtained by calculating, for the given position of loads, the moment at D for a beam $A B$ and twice the moment at K for a beam $C D$ and adding the results. This applies to both uniform and concentrated load systems. This graphic analysis brings out clearly that the effect of the trussing $E H F$, in adding to the stress in $E F$, is the same as that of a trussed beam $E H F$.

Member $E H$.—This member is the only web member in the panel $C K$ and hence its stress is determined according to Art. 158. For a load between B and D the stress in $E H$ is the same as in $E D$ of the

panel $C D$ ($H K$ and $H F = 0$); and likewise for a load between A and C. The lines $B' D''$ and $A' C''$ are then drawn as in Art. 158. From D to K the stress follows the law of the line $B' D''$, hence the influence line is completed as shown in the figure. If G_1 = load on $C K$ the criterion is $\frac{G_1}{N K'} = \frac{G}{N B'}$, as in Art. 138. For minimum stress, if tensile, the portion $A K$ is loaded, the criterion being given by the influence line $A' C'' N$. If this position gives a stress which, combined with the dead-load stress, is compressive then the counter $C H$ is required.

In calculating the maximum stress the reaction and joint load at C are found and then the stress in $E H$ by any of the methods applicable to trusses with inclined chords.

Member $E C$.—For maximum compression in $E C$ the loads are in about the same position as for maximum tension in $E H$. There will be no live-load stress in $T E$ and hence $E C$ can be treated exactly as $E H$.

Member $H D$.—The stress in this member will not be changed if the separate small truss $E H F$ [Fig. (*a*)]. be substituted for the construction as given. This being done it is seen that the load at K is distributed to E and F in the same manner as if it were applied to a beam or stringer reaching from E to F. The effect on $E D$ is the same as if applied at the lower chord on a beam or stringer from C to D. The stress in $E D$, or $H D$ of the actual truss, will therefore be calculated as if the secondary members, $H K$ and $H F$, did not exist. The influence line is shown in Fig. 41, and the calculations are the same as explained in Art. 158.

The Counter Members $C H$ *and* $H F$.—There remain to be considered the members $C H$ and $H F$ as counters; load headed towards the right. All diagonals in the panel $C D$ are designed and built as tension members and cannot carry compression except as may be possible by means of initial tension produced in adjustment. It will be assumed, therefore, as in the usual case of tension diagonals, that in every panel the web stress is carried by that member which will be stressed in tension.

Consider first the member $H F$. When this receives its maximum counter stress, $H D$ will be relieved and the member $E H$ will support

the point H. The influence line for $H F$ is then similar to that for $E H$, $C' C''$ being the stress in $H F$ for load at C and $D' D''$ for load at D. The complete line is $A' K'' D'' B'$. The stresses are readily found as explained in Chap. IV. The criterion for maximum stress is given by the influence line $A' K'' N$, and is $\frac{G_1}{K' N} = \frac{G}{A' N}$.

Member C H.—When this is a maximum, $H F$ will also be stressed and $H D$ will be relieved. On account of the small angle between $C H$ and $H F$ the stresses in these two members ($H K$ not acting), will not be quite the same but their ratio will be constant. (The angle $E H C$ being 90° the stress in $C H$ = stress in $H F \times \cos \beta$, Fig. 41 (*b*). Hence for loads outside of the panel $C D$ the influence line for $H F$ may be used also for $C H$. For loads on $C D$ the effect of the hanger $K H$ is the same as if held by the separate small truss, and hence, as shown for $H D$, we may neglect the sub-panelling and consider the main panel $C D$, giving the influence line $A' C'' D'' B'$, from which the criterion may be written. The stress for any given position is readily calculated.

Maximum Tension in E C.—In this form of truss there will be little or no dead-load tension in $E C$, owing to the effect of the sub-tie $T E$, and the dead load applied at E. The live-load tension is found as in Art. 160, by using the influence line for $E H$ and placing the live load so as to cause the resultant stress in this member to be zero, then calculate the stress in $E C$. With zero stress in $E H$ ($C H$ acting), the influence line for tension in $E C$ is as shown in Fig. 41 (*c*). In the case at hand a live load placed from A to X so as to cause zero stress in $E H$ will load joint L so that the resulting tension in $E C$ will be little or nothing.

164. Double Intersection Trusses.—*The Whipple Truss.*—The influence line for either chord or web stress in a truss with a double system of bracing is made up of straight lines of many different inclinations. Since the criteria for maximum values contain as many terms as there are different inclinations in the influence lines, in this case they are very difficult of application.

For example, take the Whipple truss of Fig. 42, and consider the chord member $F H$. The centre of moments for this member for loads on the full system is at C. For loads at the joints of this system

the ordinates for moment at C are ordinates to the influence line $A'\ I\ B'$. While for loads on the dotted system, the centre of moments for $F\ H$ is at D, and the ordinates for moment at D are ordinates to $A'\ K\ B'$. Hence the influence line for stress in $F\ H$ is the broken line $A'\ a\ b\ c\ I$-$K\ d\ e.\ .\ .\ .\ B'$. The points a and i are taken half-way between

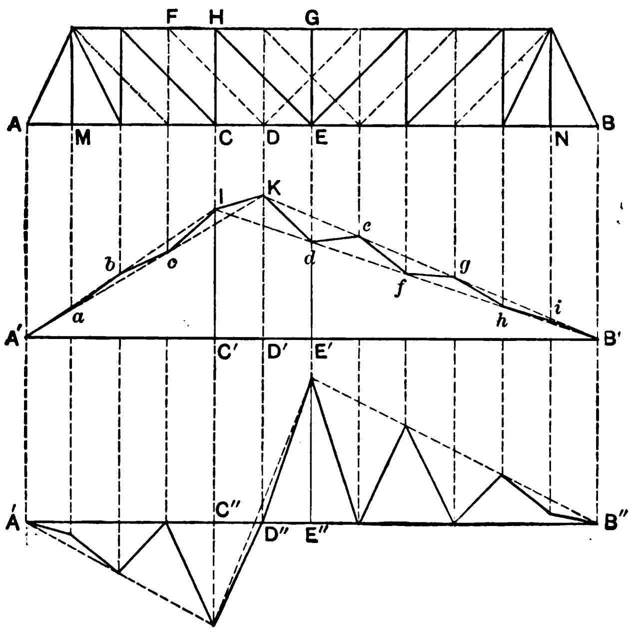


FIG. 42.

$A'\ K\ B'$ and $A'\ I\ B'$, assuming the loads at M and N to be equally divided between the two systems.

The influence line for shear in panel $C\ E$ of the full system is the full broken line of the lower figure, the ordinates to this line being zero for the loads at joints of the dotted system. Loads at M and N are equally divided.

The influence line for shear exhibits in a striking manner the effect of concentrated loads upon the web members when the distance between the loads is an even number of panel lengths; as, for example, two engine concentrations when the panel lengths are twenty-five feet, the length of a locomotive being about fifty feet. Panels of such length

are therefore, to be avoided in a double system, and a length chosen which will bring the concentrations upon different systems.

The cumulative effect upon chord members is seen to be very small.

In calculating stresses in a double-intersection truss influence lines may be used directly or an equivalent uniform load employed. The latter method will be satisfactory for chords, but where the concentra-

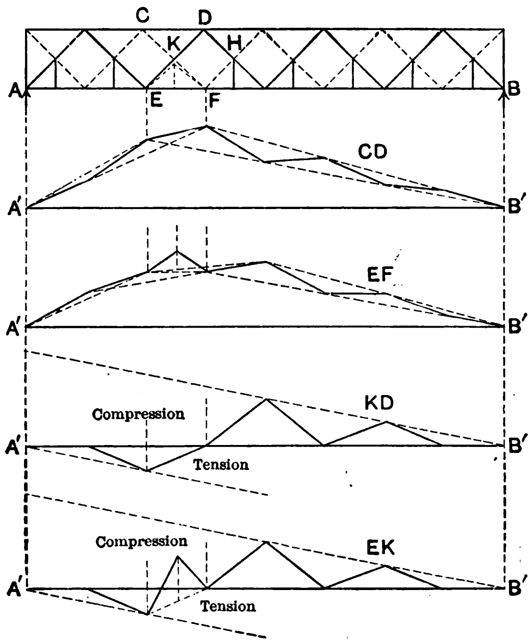


FIG. 43.

tions are heavy and widely spaced, as would be the case to a large degree with heavy coal or ore cars, the uniform load method would not give reliable results for web stress; the influence lines should be used.

165. *The Subdivided Double Triangular Truss.*—Fig. 43 shows such a truss and the influence lines for two chord and two web members. The student should be able to follow the construction. In such a case the influence lines are very useful even for uniform loads.

Section VII.—Skew-Bridges

166. Influence Lines.—Fig. 44 illustrates a skew-bridge similar to that shown in Art. 109. As there assumed it may be considered that the loads are applied along the centre lines *a b*.

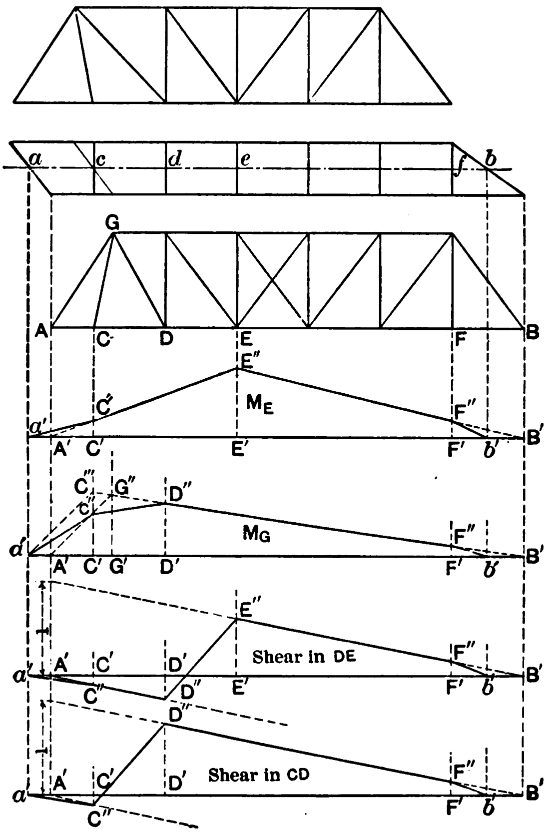


Fig. 44.

Moment at E.—For a load placed at any point between *c* and *f* the joint loads on the truss *A B* will be the same as an ordinary truss of span *A B*, and hence the moment at *E* will be the same. The influence line from *c* to *f* will therefore be the usual influence line for bending

moment, $A' E'' B'$, in which $E' E'' = 1 \times \frac{A E \times E B}{A B}$. For a load between c and a the moment will vary from a value $C' C''$ for a load at c to zero for load at a; and for a load between f and b the moment varies as the straight line $F'' b'$ thus giving the complete influence line $a' C'' E'' F'' b'$.

From this influence line the criterion for maximum moment may be written out. It is seen, however, that the true influence line differs but slightly from $A' E'' B'$, and as regards the *position* of the loads, it may be assumed the same. Hence for a maximum moment at F the average unit load on length $A E$ must be equal to that on length $A B$.

Moment at G.—The stress in $C D$ is found from bending moment at G. The ordinates at C', D', and F' are found from the influence line for moment at G in a beam $A B$, as in the Warren truss, Art. 130.

$$G' G'' = 1 \times \frac{A' G' \times G' B'}{A' B'}.$$

The influence line is completed by drawing the straight lines $a' C''$, $C'' D''$, and $F'' b'$. The criterion for moment at D' in a beam $a' B'$ may be used as an approximate rule for determining position of loads.

The stress in $A C$ is also found from moments at G, but note that the joint load at C is on the right of the section. The influence line for this moment will be obtained in the figure by continuing the line $F'' G''$ to C''' and drawing the line $a' C'''$. Place the loads as for moment at C' in a beam $a' B'$.

Shear in Panel D E.—Construct first the line for shear as in an ordinary truss $A B$. It is $A' D'' E'' B'$. This is then modified by drawing the lines $a' C''$ and $F'' b'$, as for moment. The position of loads may be taken as for a truss $A B$.

Shear in Panel C D.—The ordinates at C', D' and F' are determined as for shear in truss AB. The lines a' C'' and $F'' b'$ complete the line.

167. Calculation of Stresses.—For uniform loads the stresses are given by the areas of the respective influence lines. For concentrated loads the approximate criteria as above noted may be used for position of loads. The stresses should then be correctly calculated for the actual truss. All loads between c and f may be treated in the same way as

in a square truss A B, but loads on a c and f b are best treated by calculating the actual floor-beam reactions at c and f due to these loads and then applying one-half of each floor-beam load to joints C and F. The stresses are then determined by the usual methods.

SECTION VIII.—CONVENTIONAL LOAD SYSTEMS

168. The Train Load.—The train load generally assumed as the basis of stress calculations consists of two locomotives coupled in a direct position and followed by a uniform load representing the train load. For double-track structures the loading is generally double. A great variety of engine loadings has been used by the various railroad companies, each road, as a rule, selecting a loading based on the weights of the heaviest locomotive in service or to be anticipated in the near future. Such great variety in specifications for loads has made the calculation of stresses by exact methods a somewhat troublesome question, and has led to numerous proposed conventional systems of loads which will give approximately the same results and whereby the calculations may be much simplified.

These conventional loads are of two kinds: (*a*) systems of standard locomotive loads agreeing approximately with actual loads but in which the wheel weight and spacing are modified to secure greater simplicity; (*b*) systems in which wheel loads are not used at all, but some simpler "equivalent" loads. Of the first class the loads proposed by Mr. Theodore Cooper are the only ones that have come into extensive use. Of the latter a large number have been proposed, and several have been used to a considerable extent. Mr. J. A. L. Waddell has proposed a system of standard wheel loads, and at the same time has derived sets of equivalent uniform loads corresponding thereto.*

169. Cooper's Conventional Wheel Loads.—The loads proposed by Mr. Theodore Cooper in 1894 and now used by a large number of railroads consist of a series of standard loadings, any one of which may be obtained from any other by the use of a constant multiplier. The wheel spacing is the same in all. One of these loadings is tabulated in Art. 140.

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* The Compromise Standard System of Live Loads for Railway Bridges and Equivalents for same. By J. A. L. Waddell.

It consists of two consolidation engines with 50,000-lb. axle weights, followed by a train weighing 5,000 lbs. per foot. This loading is called "Class *E*–50." Other loadings are known as *E*–40, *E*–45, *E*–60, etc. In E–40 loading, for example, the axle weights are 40,000 lbs., and the

MAXIMUM MOMENTS, SHEARS, AND FLOOR-BEAM CONCENTRATIONS FOR COOPER'S E–50 LOADING (1,000-POUND UNITS).

For one rail.

Span, Ft.	Max. Mom. near Centre.	Max. End Shear.	Max. Floor-beam Load.	Span, Ft.	Max. Mom. near Centre.	Max. End Shear.
10	70.3	37.5	50.0	46	1036.9	103.5
11	82.1	40.9	54.6	48	1110.0	106.0
12	100.0	43.7	58.3	50	1188.8	108.9
13	118.7	46.1	61.6	52	1268.7	111.6
14	137.5	48.2	65.2	54	1351.2	114.0
15	156.2	50.0	68.3	56	1440.0	116.2
16	175.0	53.1	71.1	58	1528.6	119.2
17	193.7	55.9	73.5	60	1624.5	122.0
18	212.5	58.4	75.8	62	1720.5	125.1
19	233.3	60.5	78.6	64	1819.4	128.3
20	257.8	62.5	81.9	66	1924.2	131.2
21	282.5	64.3	85.0	68	2029.5	134.7
22	307.1	65.9	87.7	70	2134.3	138.1
23	331.8	67.4	90.2	72	2240.0	141.7
24	356.5	69.3	92.5	74	2348.8	145.4
25	381.3	71.0	94.6	76	2463.7	148.8
26	406.0	72.6	97.1	78	2580.5	152.1
27	430.8	74.1	100.1	80	2700.5	155.2
28	456.9	75.5	102.9	82	2820.5	158.6
29	484.9	76.9	105.4	84	2945.8	161.9
30	513.2	78.8	107.8	86	3074.3	165.1
31	541.1	80.5	110.6	88	3205.1	168.4
32	569.3	82.2	113.7	90	3338.0	171.5
33	597.2	83.7	116.7	92	3470.0	174.7
34	625.5	85.1	119.4	94	3607.0	177.9
35	653.8	86.5	122.0	96	3743.0	181.0
36	685.7	88.2	....	98	3883.0	184.4
37	717.9	89.9	....	100	4025.0	187.5
38	750.0	91.4	....	105	4422.0	195.1
39	783.4	92.9	....	110	4859.0	202.5
40	819.4	94.3	....	115	5306.0	209.9
42	891.9	97.6	....	120	5768.0	217.1
44	964.4	100.7	....	125	6245.0	224.3

train load is 4,000 lbs. per foot. The great advantage of such a system of loads is that stresses may be calculated for any loading and reduced to any other by multiplying throughout by a constant factor. A single moment table or diagram thus suffices for all classes. Where this

system has been adopted the class most used at the present time (1910) is E–50, with a general tendency towards still heavier loads.

170. *Moments, Shears and Floor-beam Concentrations for Cooper's E–50 Loading.*—In the preceding table are given for the E–50 loading the maximum bending moments and the maximum shears at end, for beams from 10 to 125 feet in length, and the maximum floor-beam concentration for panel lengths from 10 to 35 feet. The shears at the quarter point and at the centre may be found very closely by taking 5/8ths and 2/7ths, respectively, of the end shear.

171. Conventional Systems of Equivalent Loads.—The refinements of calculation by the wheel-load method are entirely unwarranted by the conditions under which a bridge is actually loaded. Instead of the load being stationary, as assumed, it is moving at a high rate of speed, and the stresses arising from the impact and vibration due to unbalanced locomotive drivers, inequalities of track, etc., are a large and variable percentage of the statically determined stresses. These considerations, together with the somewhat lengthy process of stress calculation by the wheel-load method, have led to a considerable use of various kinds of simplified equivalent loads. For ordinary bridges, however, there is really not much gained. The selection of such equivalent loads is somewhat troublesome. The actual loads on the bridge are the engine wheel loads, and to substitute for them some other form of loading requires a careful study of the relative results given by the two systems, as the fitness of any equivalent load method must be based finally upon its agreement with the wheel-load method. Since the introduction of Cooper's conventional engine loadings, whereby the work of calculation is greatly simplified, there is little demand for any simpler form of load for ordinary bridges. For complex structures, such as arches, cantilever bridges, etc., especially if of long span, some form of equivalent load is often desirable. For double-intersection trusses also, the wheel-load method is difficult to use, as shown in Art. 164.

Of the class of conventional loads here considered there are three that have been used to a considerable extent and deserve mention.

First. The use of two concentrated excess loads, placed fifty feet apart, which may occupy any position in the uniform train load, and which may be conceived as rolling across the span on top of the train

load. They would be near the head of the train for shears, and one of them at the joint in question for moments. This is a fair substitute for wheel concentrations on double-intersection trusses, as it brings out the effect of the driver concentrations upon the separate web systems.

Second. The use of one such concentrated excess load in place of two, it always being placed at the joint in question, while the train load covers the whole span for maximum moments, and reaches to a particular point in the panel in question for maximum shears. This method has been used to a considerable extent. For moments it gives the same results as the use of a uniform load; for shears or web stresses the results are nearly correct for the panels near the end, but are considerably too high for those near the centre.*

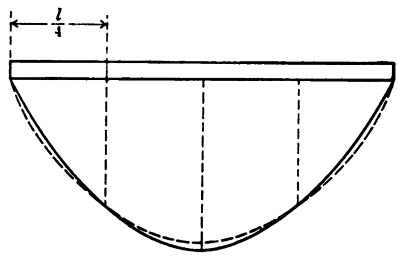


FIG. 45.

Third. The use of an equivalent uniform load. This is the simplest possible form of equivalent load, and this fact has led to many attempts to secure its general adoption. For short girder spans its accuracy is hardly sufficient, nor is its use of much advantage since a table such as that given in the preceding article is all that is necessary with the wheel-load method. For spans over 100 feet in length it will give results accurate to within two or three per cent. for most members if the load is properly selected, and for such spans it furnishes a desirable substitute for the wheel-load method. For very long spans and for complex structures it is much preferable to the wheel-load method. Methods of determining the proper equivalent load will now be considered.

172. The Single Equivalent Uniform Load.—The moment diagram for a uniform load acting upon a simple beam is a parabola; also the curve of maximum moments in a beam due to a single moving load is a parabola (Art. 66). For a series of concentrated loads the moment curve varies only a little from a parabola, the actual curve from the usual engine loading being a little flattened at the centre and having a

* For a full discussion of this system see Trans. Am. Soc. C. E., Vol. XV, p. 474, Vol. XXI, p. 575.

little sharper curvature towards the ends of the beam as compared to a parabola. If a parabola be drawn having the same average ordinates as the true moment curve, the two curves will cross at about the quarter-point. In Fig. 45 the dotted line represents the curve of maximum moments, and the full line the parabola. From these considerations it is seen that the uniform load which will give the same average moments as the wheel loads is such a load as will give the same moment near the quarter-points. To determine such load we may then calculate the actual maximum moment at the quarter-point, place it equal to the moment caused by a uniform load of p per foot, and solve for p.

For a uniform load p per foot the moment at the quarter-point in a beam of length l is equal to $3/32\, p\, l^2$. If $M_{\frac{1}{4}}$ is the actual maximum moment, then we have

$$\frac{3}{32}\, p\, l^2 = M_{\frac{1}{4}},$$

whence

$$p = \frac{32}{3}\,\frac{M_{\frac{1}{4}}}{l^2}. \qquad \qquad (1)$$

This gives a single uniform load which will give nearly the same moments in a beam or truss as the given wheel-load system. For points near the end the resulting moments will be somewhat too small, while near the centre they will be too large. At the quarter-point they will necessarily be correct since they have been made equal at this point.

For shears or web stresses the use of the same load as for moments gives too small results, as the effect of the heavy concentrations is relatively greater for a partially loaded bridge. Fair results may be obtained by the use of the same loading if the conventional method of shear calculation is used as explained in Art. 82 (*a*). Still better results may be obtained by using the equivalent load for moments, as found by eq. (1), for a length of span equal to the *loaded* length of bridge.

173. An Exact Equivalent Uniform Load System.—(*a*) *Equivalent Load for Moment.*—In Art. 172 a single equivalent uniform load was determined by equating moments at the quarter-point. Taking the actual concentrated load stresses as a basis, such a load necessarily gives exact results at the quarter-point but only approximate results at other points. Exact results can evidently be obtained for any moment centre if the equivalent load used for such centre be obtained in the

manner above described for the quarter-point. Thus a uniform load determined for the 1/8th point will give correct values for moment at that point, and another load determined for the centre will give correct results for the centre. Each moment centre will have a different equivalent load.

(*b*) *Equivalent Load for Shears.*—Consider the shear in panel *C D* of an 8-panel truss. The influence line is shown in Fig. 46. The

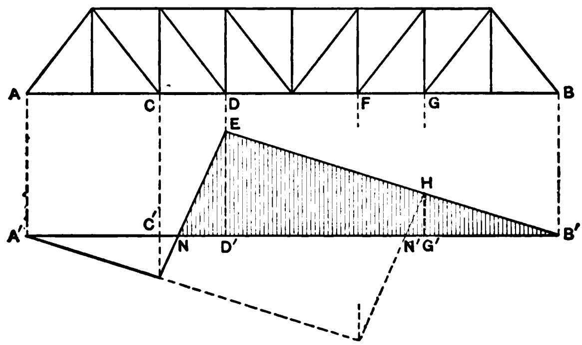


FIG. 46.

portion *N B′* will be loaded for maximum shear and the criterion for shear is the same as that for moment at *D′* in a beam *N B′*. It also follows that the value of the shear itself is proportional to such moment. Hence the equivalent uniform load for shear in *C D* will be the same as the equivalent uniform load for moment at *D′* in a beam *N B′*. In an 8-panel truss the ratio *N D′* to *N B′* is 1/8 for all panels, hence the correct equivalent load for shear in panel *C D* will be the equivalent load for moment at the 1/8th point in a beam of length *N B′*. The shear itself will equal this equivalent uniform load per foot, multiplied by the area *N E B′* of the influence diagram.

For any other panel, as *F G*, the equivalent load is that for moment at *G′* in a beam *N′ B′* (*N′ G′*/*N′ B′* = 1/8). The shear = load per foot × area *N′ H B′*.

Thus it is seen that the same system of equivalent loads can be used for shears as for moments by selecting the load in accordance with the form of the influence line.

174. Equivalent Uniform Loads for Cooper's E–50 Loading.— It is evident that a set of equivalent uniform loads, such as described

in the preceding article, is of no assistance in the calculation of one or two trusses, as the concentrated load stresses are required as a basis for determining the equivalent uniform loads; but it is possible to tabulate or diagram the equivalent uniform loads for any particular engine loading, such as Cooper's standard loading, for a wide range of span lengths, so that the equivalent load for any desired moment or stress can be found at once.

Fig. 47 is such a diagram of equivalent uniform loads for Cooper's E–50 loading, for span lengths from 40 to 500 feet. The diagram gives directly the proper load for moments for the 1/10th, 2/10ths, 3/10ths, 4/10ths, and centre points, for all the given span lengths; for intermediate points the load may be found closely by interpolation. The diagram is plotted from calculations of moments from the actual concentrations and the uniform loads are those which will give the same moments as the concentrations. The loads are therefore an exact equivalent, and are approximate only to the extent which the interpolations may be approximate.

175. *Advantages of such an Equivalent Load System.*—Such a diagram as given in Fig. 47 is quite accurate enough for stress calculations, or it may be used for checking calculations made by other methods. Used in connection with influence lines constructed graphically, the system is rapid and easy of application. It is especially applicable for any system of loading frequently used, such as Cooper's loadings, or standards used in the bridge department of a railroad company. Constructed for several loadings it shows very plainly the relative influence of the different systems on the various parts of a structure.

Thus, for example, it is noted in Fig. 47 that even for a 500-foot span the equivalent load for centre is considerably less than for points near the end. This is due to the fact that the train load of Cooper's standard is considerably lighter than the engine load. With a heavier train load the several lines on the diagram would be closer together. Note also the crossing of some of the lines, at span lengths from 150 to 200 feet. This is explained by the fact that for such spans the moments near the centre are relatively high, the heavy driver concentrations of the second engine coming near the centre of the span. On the contrary, for spans of 75 to 100 ft. the loads near the centre are light and the centre moments are relatively small.

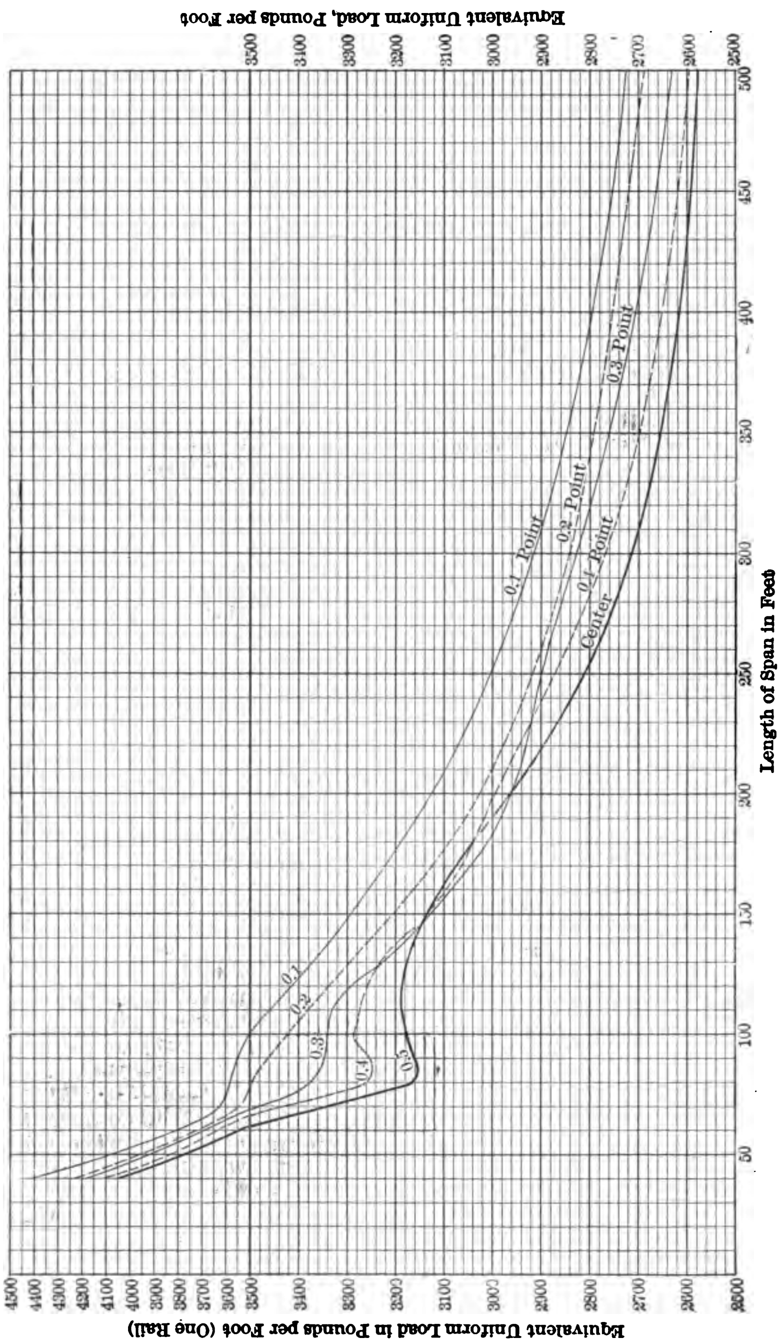
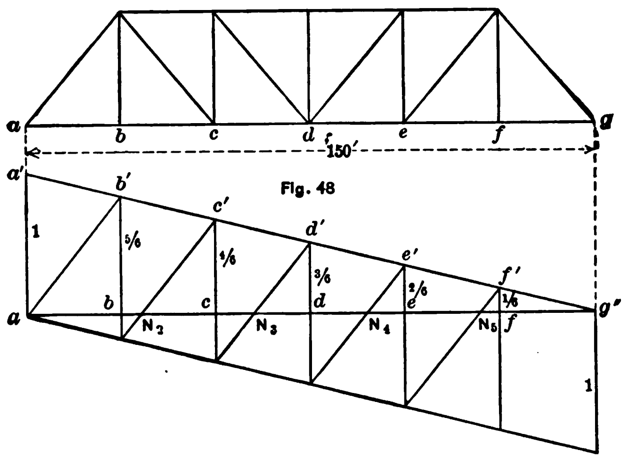


FIG. 47.—Equivalent Uniform Loads for Cooper's E-50 Loading.

A different spacing or arrangement of loads would modify the curves very materially; and if a system of equivalent uniform loads was to be adopted as a standard the curves should be "smoothed out" somewhat, as more likely to represent actual conditions. However, a set of such diagrams, constructed for some of the heaviest locomotives in use, would enable a proper standard equivalent loading to be easily selected.

176. *Application to a Pratt Truss.*—Required the moments and shears in the Pratt truss of Fig. 48.

Moments.—M_b. For moment at b the equivalent load is that for the 1/6th (or 0.17) point in a 150-ft. span. Interpolating in Fig. 47



FIGS. 48 AND 49.

this is found to be 3,240 lbs. per ft. Then by eq. (2), Art. 78, $M = \frac{1}{2} p\, d^2\, m\, m' = \frac{1}{2} \times 3{,}240 \times 25^2 \times 1 \times 5 = 5{,}060{,}000$ ft.-lbs.

M_c. Point c is the 2/6th (0.33) point, and from Fig. 47, $p = 3{,}130$. Hence $M_c = \frac{1}{2} \times 3{,}130 \times 625 \times (2 \times 4) = 7{,}820{,}000$ ft.-lbs.

M_d. For the centre point of a 150-ft. span, $p = 3{,}135$; and $M_d = \frac{1}{2} \times 3{,}135 \times 625 \times (3 \times 3) = 8{,}805{,}000$ ft.-lbs.

Shears.—The influence lines for all panels are shown in Fig. 49, together with the values of the ordinates at the panel points.

Panel $a\,b$.—The influence line is $a\,b'\,g'$ and the equivalent load is that for moment at b in the span $a\,g'$, which is already found to be

3,240 lbs. per ft. Then shear in $a\,b$ = 3,240 × area $a\,b'\,g$ = 3,240 (½ × 5/6 × 150) = 202,500 lbs.

Panel $b\,c$.—The equivalent load is for moment at c in the beam $N_2\,g'$. The ratio of $\overline{N_2\,c}$: $\overline{N_2\,g'}$ = 1/6 and $N_2\,g'$ = 4/5 × 150 = 120 ft. Hence the proper load is the equivalent load at the 1/6th point in a 120-ft. span. This is 3,370 lbs. per foot; whence, shear is $b\,c$ = 3,370 × area $N_2\,c'\,g'$ = 3,370 × (½ × 4/6 × 120) = 134,800 lbs.

Panel $c\,d$.—$N_3\,g'$ = 90 ft. Load for 1/6th point in a 90-ft. span = 3,490, and shear = 3,490 (½ × 3/6 × 90) = 78,500 lbs.

Panel $d\,e$.—$N_4\,g'$ = 60 ft., p = 3,750. Shear = 3,750 × (½ × 2/6 × 60) = 37,500 lbs.

Panel $e\,f$.—$N_5\,g'$ = 30 ft., p = about 4,500 lbs. Shear = 4,500 (½ × 1/6 × 30) = 11,200 lbs. (The diagram has not been extended to include spans less than 40 feet, as the calculations for such spans are of little importance except for plate girders. For these the table of Art. 170 gives all needed information.)

Comparing these results with those obtained in Arts. 142 and 143, from the actual concentrations, we have the following:

Moments

	From Concentrations	From Equivalent Loads
M_b	5,058,000	5,060,000
M_c	7,794,000	7,820,000
M_d	8,805,000	8,820,000

Shears

	From Concentrations	From Equivalent Loads
$a\,b$	202,300	202,500
$b\,c$	134,300	134,800
$c\,d$	78,600	78,500
$d\,e$	37,100	37,500
$e\,f$	10,500	11,200

177. *Application to a Curved-Chord Pratt Truss.*—Required the stresses in $E\,D$, $E\,C$, and $H\,K$ of Fig. 50. These have been found by the concentrated load method in Art. 160.

The influence lines for these members are shown in the figure.

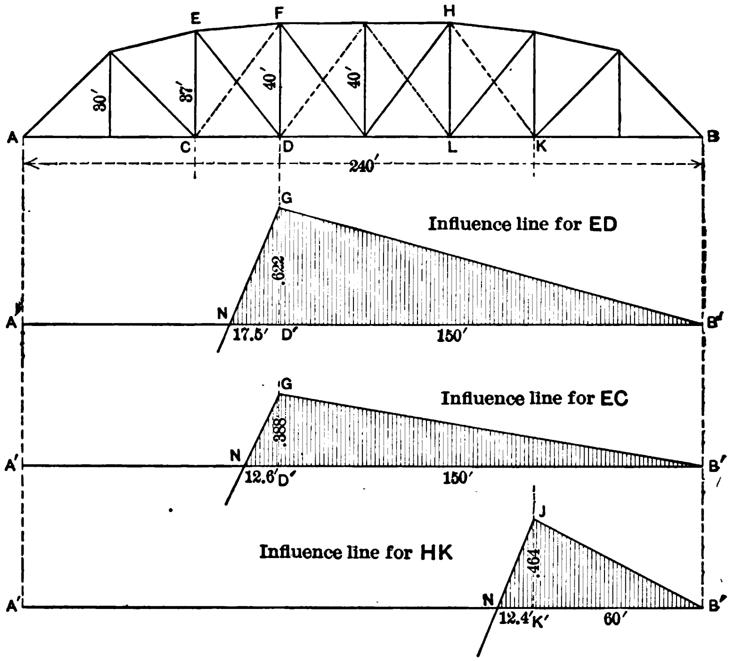


FIG. 50.

They are conveniently constructed by the graphical method explained in Art. 158. The dimensions shown in the figure are all that are needed in the calculations.

Member $E\,D$.—The ratio of $N\,D'$ to $N\,B' = 17.5/167.5 = 0.105$. Hence the equivalent load is that for the 0.105 point in a beam of 167.5 ft. span. Fig. 47 gives this at 3,220 lbs. per foot. Hence stress in $E\,D = 3{,}220 \times$ area $N\,G\,B' = 3{,}220\ (\frac{1}{2} \times 0.622 \times 167.5) =$ 168,000 lbs.

Member $E\,C$.—In a similar manner we find $p = 3{,}270$ and stress $= 3{,}270\ (\frac{1}{2} \times 0.388 \times 162.6) = 103{,}200$ lbs.

Member $H\,K$.—The value of $p = 3{,}540$ and stress $= 3{,}540\ (\frac{1}{2} \times 0.464 \times 72.4) = 59{,}500$ lbs.

The stresses obtained in Art. 160 were, respectively, 168,100; 102,500; and 58,700 lbs.

178. General Applicability of the Exact Equivalent Uniform Load System.—The equivalent uniform load system explained in the preceding articles is strictly applicable to all cases where the influence line of the function, or that part of the influence line which is utilized, is of the form shown in Fig. 51; that is if it is triangular in form. The system is, therefore, strictly applicable to all members but one of the Pettit truss (Fig. 41), and to all single intersection trusses, excepting the moments at points in the unloaded chord of a Warren or similar truss. For such cases a sufficiently close approximation to the proper load may be obtained by substituting for the true influence line, two straight lines drawn so as to give about the same average ordinates. The load being determined, the stress is equal to this load multiplied by the area within the true influence line. The only approximation involved in this method is in selecting the equivalent load, an error generally quite negligible.

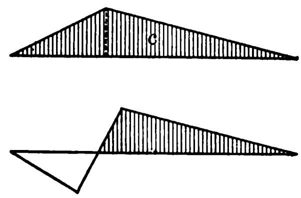


FIG. 51.

This method is also very useful in the analysis of more complex structures, such as swing bridges, arches, and cantilever bridges. In many cases, as in the cantilever bridge and the three-hinged arch, the method is strictly applicable; in other cases a very close value of the equivalent uniform load may be selected by a consideration of the form of the influence line. Examples of its use are given in Part II.

CHAPTER VI

LATERAL TRUSSES, TRESTLES AND TOWERS

179. The lateral pressure upon a bridge or roof truss arising from wind, or the centrifugal force due to loads moving in a curve, is resisted by means of horizontal trusses generally called "lateral" trusses, placed between the chords of the vertical trusses. The chords of the vertical trusses thus form the chords of the lateral trusses. Roof trusses are usually braced laterally in pairs, the end pair taking the greater part of the end pressure, the others being merely stiffened against buckling,

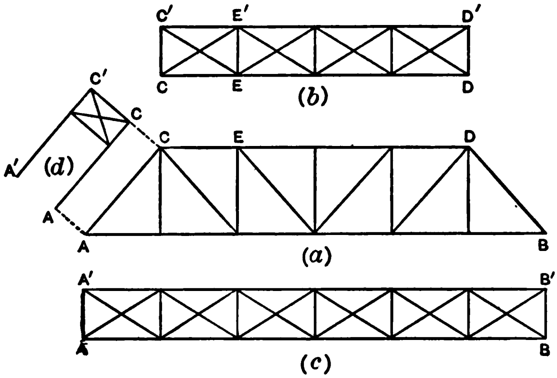


FIG. 1.

according to the judgment of the engineer. The stresses in the end lateral system are computed in the same way as are those in the lateral systems of bridge trusses, which will be discussed in detail.

180. Forms of Lateral Trusses for Bridges.—The type of trussing adopted for lateral trusses varies with the size and type of the main structure. Fig. 1 illustrates the usual lateral system of a through Pratt truss. The upper laterals are shown in Fig. (*b*), and the lower laterals in Fig. (*c*). The wind pressure, or other external load, acting upon the upper lateral truss, is carried by that truss to points $C\,C'$ and $D\,D'$

which act as abutments. The loads so transferred are carried by the end or "portal" bracing, shown in Fig. (*d*), to the abutments at A and A'. Loads on the lower lateral system are carried directly to the abutments at $A\ A'$ and $B\ B'$.

In the through-bridge the end bracing is incomplete, an arrangement which requires the members $A'\ C'$ and $A\ C$ to act as beams. In the deck structure the end bracing consists of full diagonal bracing.

The diagonals are often made of rods designed to act as tension members only, but in the best modern practice they are usually made of forms capable of resisting compressive stresses (rigid bracing). However, unless the unsupported lengths of the diagonals are comparatively short they should be assumed to act in tension only. Where they are sufficiently well supported to act as effective compression members, then the lateral shear may be assumed to be carried equally by the two diagonals, one-half in tension and one-half in compression. An example of well-supported diagonals is the lower lateral system of a through-bridge where the diagonals are attached to the stringers and to each other at the several points of intersection. Their design is fully considered in Part III.

In the upper lateral system $C\ C'$, $E\ E'$, etc., are compression members and are called "lateral struts." Where the diagonals act as tension members only, these struts act as the verticals of the Pratt truss which the lateral system then becomes. In the lower lateral system the floor beams act as the lateral struts. In the case of a deck-bridge the conditions are reversed.

The type of lateral truss shown in Fig. 1 is generally used for all long-span steel bridges of whatever form. In Howe trusses the lateral system is generally also made of the Howe type; that is, the diagonals are wooden struts and the perpendicular members are rods.

In the case of small structures, the diagonal length is not great and a single diagonal system of bracing of the Warren type is often used. This is the type generally employed for the laterals of girder bridges, and, frequently, for the upper laterals of short through-bridges.

181. Lateral Systems Necessary for Complete Bracing.—Considering the entire bridge as a framed structure in space it will be found that, in addition to the main vertical trusses, lateral trusses placed along both top and bottom chords, together with the portal or end bracing, make

the system of framework complete and give a rigid structure. For sake of increased lateral stiffness, however, it is desirable in all large bridges to place additional transverse bracing at each panel point between each pair of vertical posts. This bracing in through-bridges is made of the same form as the portal bracing.

Often in short-span, deck-girder bridges only one lateral truss system is used, that along the top flange, in which case transverse bracing is depended upon to support and stiffen the lower flange. The end cross-bracing then carries the accumulated load to the abutments. In pony trusses and through-girder bridges the lateral pressure at the top is resisted by the rigidity of the vertical posts or by the web with its stiffeners and gusset plates.

182. The Wind Pressure.—The wind pressure is assumed to act at right angles to the structure and to be concentrated at joints by the members of the truss and floor system in the same manner as the loads on the vertical trusses. The wind loads are commonly figured on the basis of about 50 lbs. per square foot on the unloaded structure and about 30 lbs. per square foot on the loaded structure and its load, the exposed area of the bridge being taken as the exposed surface of all trusses and floor as seen in elevation. The wind pressure acting upon the live load is considered as a moving load, while that acting upon the bridge is generally treated as a fixed load although sometimes this also is considered a moving or variable load. (For a discussion of the amount of pressure caused by wind, see Art. 68, of Chapter III.)

The pressure upon the upper half of the truss is assumed to be taken by the upper laterals, and that upon the lower half by the lower laterals. The pressure upon the load is assumed to be all taken by the lateral system belonging to the loaded chord; but as this pressure is applied some distance above the plane of the laterals, its overturning effect must also be considered. This effect is separately discussed in Art. 190.

In addition to wind pressure, a bridge is subjected to considerable lateral forces due to vibration and the impact of moving loads, especially in the case of railway bridges; and to secure desired rigidity it is usual to specify, for all but the longer spans, lateral forces considerably in excess of those which would be obtained by the use of the unit pressures above mentioned. Thus Cooper specifies for highway bridges a pressure of 150 lbs. per linear foot for the laterals of the unloaded chord

and 300 lbs. per foot for those of the loaded chord, 150 lbs. of the latter to be treated as a moving load; for spans exceeding 300 feet, add to each, 10 lbs. per foot for each additional 30 feet. For railway bridges he specifies 150 lbs. for the bracing of the unloaded chord and 600 lbs. for the loaded chord, 450 lbs. of the latter to be treated as a moving load and as applied at a distance of 6 feet above base of rail. For the unloaded chord the same increase for long spans is provided as in highway bridges.

The specifications of the Maintenance of Way Association require that all spans shall be designed for a lateral force on the loaded chord of 200 lbs. per linear foot plus 10 per cent. of the specified train load on one track, and 200 lbs. per linear foot on the unloaded chord; these forces being considered as moving.

183. Stresses in the Lateral Trusses Due to Wind Pressure.—With the load per foot known the stresses are readily found by the methods of Chapter IV. The loads on the lateral system of the loaded chord should be assumed as all applied on the windward side; those on the lateral system of the unloaded chord may be assumed as applied equally on the two sides, although to assume them all applied on the windward side is sufficiently accurate, as the only effect would be to increase the stress in each strut by one-half panel load, a matter of no practical consequence.

Where the diagonals are tension members, only one set is assumed to act for a given direction of wind pressure. Counter-stresses need not be calculated, as the reversal of wind pressure gives greater stress in the web members concerned than any partial loading. If the diagonals are assumed to take compression as well as tension, then the two diagonals of a panel may be assumed as equally stressed. This gives in effect a double Warren system, the struts in this case serving merely to equalize the joint loads on the two systems.

The chord stresses of the lateral trusses must be combined with the stresses in the chords of the vertical trusses due to dead and live loads. The chord members also receive some stress from the "overturning effect" mentioned in Art. 182, and a still further amount from the action of the portal bracing, as explained later; so that the total wind stresses may add a very large percentage to the dead- and live-load stresses, or they may act to reverse those stresses.

Where the diagonals are assumed to take both tension and compression the chord stresses are found by taking moments at a section through the middle of the panel. Thus in Fig. 2 a section, q, is taken through the intersection of the two diagonals, and moments taken about O to get the stress S_1. In this equation the moments of S_2 and S_3 balance each other, as the two stresses are numerically equal. Hence if M_0 = bending moment at O, it follows that, $S_1 = M_0/b$. Also $S_4 = S_1$.

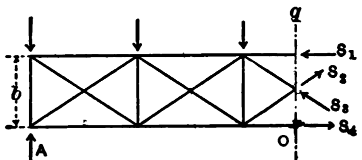


FIG. 2.

184. *Lateral Trusses for Bridges with Inclined Chords.*—In this case the lateral system belonging to the chord which is inclined does not lie in one plane, but in several planes. The exact determination of all the wind stresses is a matter of considerable difficulty, but the stresses in the lateral members themselves will be correctly determined by considering the truss flattened out into one plane and calculating the stresses as for the ordinary case. The panel lengths will not be equal, but will be the same as the actual lengths of the chord segments. The joint loads, however, may still be taken as equal and obtained by multiplying the specified wind load per foot by the horizontal panel length. The chord stresses resulting from this method of analysis will be somewhat in error, and there will also be certain stresses in the web members of the vertical truss that will not be determined. These errors and omissions are of no practical importance. This general problem is treated in Part II.

185. Stresses in Portal Bracing.—The wind pressure against the upper chord of a through-bridge is carried by the upper laterals as far as the end joints of the upper chord. From these it must be transferred to the abutment by means of the portal bracing in the plane of the end posts. Several forms of portal bracing will be analyzed. In all cases the end posts themselves constitute an essential part of the bracing.

(*a*) *The Plate-girder Portal.*—Let Fig. 3 represent such a portal projected on a plane parallel to the plane of the end posts. Let c be the actual length of the end posts, b the width centre to centre of trusses, and e the effective depth of portal beam. The load brought to the portal by the upper laterals may be considered as all applied at the windward side C. In addition to this load, there is a panel wind load

acting on the joints C and C', but which for simplicity will be assumed as all applied at C. The total load at C will then equal the shear in the end panel of the lateral truss, plus one panel wind load. Call this total load P. The only other external forces acting in the plane of the portal are the reactions at A and A'. Let these be represented as shown. The end posts will for the present be assumed as hinged at A and A'; the question of fixed ends is discussed in Art. 186.

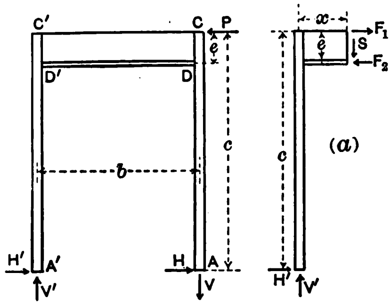


FIG. 3.

The reactions, being four in number, cannot be exactly determined by statics. It is customary, and nearly correct, to assume $H = H'$, whence we readily get

$$H = H' = \frac{P}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$V = V' = P\frac{c}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When we come to consider the stresses produced in the portal by these external forces the structure should be thought of as an irregular continuous beam, $A' \ C' \ C \ A$, made up of three beams rigidly connected.

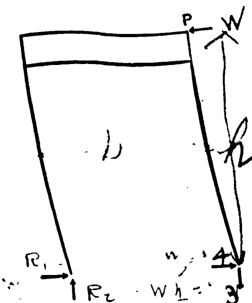


FIG. 4.

The nature of the stresses will be understood from an inspection of Fig. 4, which represents in an exaggerated manner the distorted structure. Each of the end posts will be subjected to certain bending moments and shears besides the direct stress V or V'. The portal girder will likewise have moments, shears, and direct stresses. It has a point of inflection at the centre.

The bending moments in the posts will be a maximum at D and D' (Fig. 3), and at those points will be equal to

$$M = H \times (c - e) = \frac{P}{2}(c - e). \quad . \quad . \quad . \quad . \quad (3)$$

$R_1 + R_{11} = W$ $\qquad R_2 = \frac{W}{b}$

$2R_1 = W$

The shear in each post is equal to $H = \frac{P}{2}$, and the direct stresses are $V' = V = P\frac{c}{b}$, tension in AC and compression in $A'C'$. The maximum fibre stress will occur on the right side of the leeward post at D', where the stress due to bending moment, the stress due to V', and that due to dead and live loads on the main truss are all compressive. Each post must then be designed to resist this stress. The shears in the posts are not often needed in designing, but it is important to know that they exist.

To find the stresses in the girder pass a section at a distance x from C' and consider the structure on the left, Fig. 3 (*a*). Let F_1 and F_2 be the flange stresses and S the shear, assuming the moment carried entirely by the flanges. Then with moment centre at the upper flange we derive the value

$$F_2 = \frac{H'c - V'x}{e} = \frac{P\left(\frac{c}{2} - \frac{c}{b}x\right)}{e}. \quad . \quad . \quad . \quad (4)$$

For $x = 0, F_2 = \frac{Pc}{2e}$; for $x = \frac{b}{2}, F_2 = 0$; and for $x = b$, $F_2 = -\frac{Pc}{2e}$.

That is to say, the lower flange stress is zero at the centre and has a value of $\frac{Pc}{2e}$ at the ends, compression at the leeward end and tension at the windward. Likewise with moment centre at the lower flange we derive

$$F_1 = \frac{H'(c-e) - V'x}{e} = \frac{P\left(\frac{c}{2} - \frac{c}{b}x\right)}{e} - \frac{P}{2}. \quad . \quad . \quad (5)$$

This stress is maximum tensile at the left end and maximum compressive at the right, and is everywhere equal but opposite in sign to the lower flange stress, plus a *compression* of $\frac{P}{2}$. At the centre it is equal to a compression of $\frac{P}{2}$.

The value of the shear is at all points equal to

$$S = V' = P\frac{c}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The variation in moment, shear and direct stress in the various parts of the portal are clearly shown by the shaded areas in Fig. 5.

Where end floor beams are not used, a strut $A\ A'$ must be inserted to transfer a part of the load brought to A over to A'. The total load brought to A from the upper laterals and portal is equal to $\frac{P}{2}$. From the lower laterals it receives a load equal to the shear in the end panel of the lower laterals, plus the load applied directly at A; call this load P'.

At the fixed end of the span the load transmitted by the strut may be assumed to be that necessary to equalize the loads at A and A'. This will be $\frac{P'}{2}$. At the expansion end the horizontal resistance at A may be assumed as equal to the frictional resistance of the shoe on the rollers, although provision is always made against lateral movement. The stress in $A\ A'$ will then be equal to $P' + \frac{P}{2}$ minus this frictional resistance. The latter will be determined from the net reaction at A, adding together the reactions due to vertical loads, the negative reaction $V \cos \theta$ (see Art. 188) and that due to the overturning effect of the wind against the train. The most unfavorable case will probably be when the bridge is loaded with a train of minimum weight (about 1,200 lbs. per foot). The coefficient of friction may be taken at ¼.

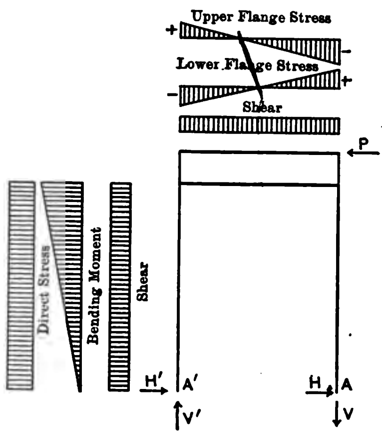


FIG. 5.

(*b*) *The Lattice Portal* (Fig. 6).—The stresses in the end posts will be the same as in the plate-girder form.

The stresses in the web members are found from the shear. As the shear is constant throughout the girder it follows that the stresses in the web members are constant and that therefore the shear on any section p is equally distributed among the web members cut. If n is the number cut by a vertical section (number of systems of bracing), then the vertical component of the stress in any web member is

$$S_v = \frac{V'}{n} = \frac{P\,c}{b\,n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Half of the members will be in tension and half in compression, but when the wind is from the opposite direction the signs of the stresses will be reversed.

To get the flange stresses pass a section p through the points of intersection of the web members (Fig. a). Take centre of moments at c. The stresses in the web members being all numerically equal and half being tensile and half compressive, the sum of their moments about any point in the vertical cd is zero. We therefore derive the same expression for F_2 as in eq. (4) for the plate-girder form. Likewise F_1 is given by eq. (5). In this case, however, x is the distance to any section p passing exactly through the points of intersection of the web members or through the centre of the panel in which the flange stresses are desired. In the form assumed the least value of x is $\frac{1}{8}\,b$ and the greatest value $\frac{7}{8}\,b$. The maximum flange stresses are therefore somewhat less than in the plate-girder form, although for a portal with four or more panels it is accurate enough to assume them the same. If there are but two panels, then the maximum stresses in the lower flange will be one-half of those in the plate-girder form, and those in the upper flange also one-half, plus the constant $\frac{1}{2}\,P$.

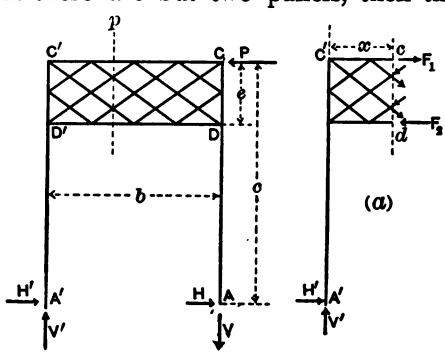


FIG. 6.

Frequently the effective depth of the plate girder or lattice portal is increased at the ends by the use of corner brackets at D and D', consisting either of simple knee-braces or of brackets with solid web. In either case the bending moments in the posts are reduced in proportion as the unsupported length is reduced. The maximum stresses in the flanges will also be somewhat reduced, but they are difficult to calculate accurately, and the best practical solution is to assume them the same as without the knee-braces. The knee-braces or bracket-flanges should then be made of about the same size as the main flanges and be well connected.

(*c*) *The Portal with Simple Diagonal Bracing* (Fig. 7).—The stresses in the posts are the same as in the previous cases, except the shears above D and D' as noted below. If the diagonals are made tension members, then $C\,C'$ and $D\,D'$ are struts. With the wind acting as shown, member $C\,D'$ is not acting. Then, passing section p, we have:

$$\text{Stress in } D\,D' = H'\frac{c}{e} = \frac{P\,c}{2\,e}. \quad (8)$$

$$\text{Stress in } C\,C' = H\,\frac{c-e}{e} + P = \frac{P\,c}{2\,e} + \frac{P}{2}. \quad (9)$$

$$\text{Stress in } C'\,D = V'.\frac{f}{e} = P\,\frac{c}{b}\cdot\frac{f}{e}. \quad (10)$$

If the diagonals are made to act as compression members only, then CC' and DD' are tension members.

(*d*) *The Portal with Knee-Braces* (Fig. 8).—This form is used where lack of head-room prevents the use of a better one. The member $C\,C'$ is a continuous beam and resists moment, shear, and direct stress. The knee-braces are subjected to direct stress only, being two-force pieces. A section cannot be passed through the posts or the member $C\,C'$ except at the ends of these members (assumed as hinged), without involving moment, shear, and direct stress among the unknowns. To avoid the difficulty pass section p through point C where there is no bending moment. Then replace stress in $E\,D$ by its horizontal and vertical components applied close to the joint D; we then have,

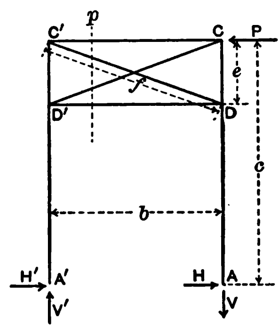


FIG. 7.

$$\text{Hor. comp. tensile stress in } E\,D = H\,\frac{c}{e},$$

whence

$$\text{Tension in } E\,D = H\frac{c}{e}\cdot\frac{f}{d} = \frac{P\,c}{2\,e}\cdot\frac{f}{d}. \quad (11)$$

The compressive stress in $E'\,D'$ is the same.

The maximum moments in the posts are at D and D' and are the same as before, also the shears and direct stresses below these points.

$$\text{Compressive stress in } C\,D = \text{vert. comp. in } E\,D - V$$

$$= \frac{P\,c}{2\,d} - \frac{P\,c}{b}. \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

This is greater than the compression in $A'\,D'$ $\left(\frac{P\,c}{b}\right)$ if d is less than $\frac{b}{4}$. The tension in $C'\,D'$ is also given by eq. (12). The shear in $C\,D = H\frac{c}{e} - H = H\left(\frac{c}{e} - 1\right)$. The shear in $C'\,D'$ is the same.

The stresses in the beam $C\,C'$ remain to be determined. The

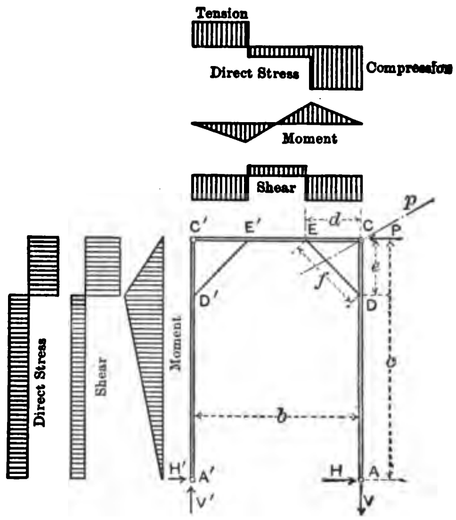


FIG. 8.

bending moment is zero at C and C', as the ends are assumed as hinged. For moment at E we have,

$$M_E = V'\,(b - d) - H'\,c = P\,c\left(\frac{1}{2} - \frac{d}{b}\right). \quad . \quad . \quad . \qquad (13)$$

The moment at E' is the same in amount but negative, and the moment at the centre is zero. For the direct stresses in $C\,C'$ we have,

$$\text{Compression in } E\,E' = P - H = \frac{P}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Then with section p and moment centre at D we have, as in eq. (9),

$$\text{Compression in } EC = \frac{H(c-e)}{e} + P = \frac{Pc}{2e} + \frac{P}{2}, \quad . \quad . \quad (15)$$

and similarly,

$$\text{Tension in } E'C' = \frac{Pc}{2e} - \frac{P}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The shear in $EE' = V'$; in EC it is equal to V' − vert. comp. in $ED = V' - H\frac{c}{d} = P\left(\frac{c}{b} - \frac{c}{2d}\right)$; in $C'E'$ it is the same as in EC.

The variations in moment, shear and direct stress are fully shown by the shaded areas of the figure.

If the knee-braces are made to meet at the centre as shown in Fig. 9, the result is a very simple and effective form of portal. The moment at E is zero, and CC' receives no bending moment or shear. Frequently the members shown by dotted lines are inserted for appearance's sake and to stiffen the members $D'E$ and ED, but they receive no definite stress. The stresses are readily obtained from the preceding equations by making $d = \frac{b}{2}$. From eq. (11) we have

$$\text{Tension in } ED = \frac{Pc}{e} \cdot \frac{f}{b}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The direct stresses in EC and ED' are given by eqs. (15) and (16).

186. *Portals with Fixed End Posts.*—In the foregoing discussion the end posts have been treated as hinged at their bases A and A'. Usually, however, they may be considered as fixed by virtue of their direct stress, due to the vertical loads, which prevents them from tipping on the pin in the plane of the portal. This will be true whenever the total stress in the end post, multiplied by one-half the distance centre to centre of bearings on the end pin, equals or exceeds the greatest bending moment which can be developed at A or A' under the assumption of fixed ends.

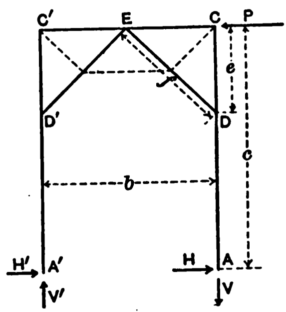


FIG. 9.

If the lower ends are fixed there will be points of inflection I and I' (Fig. 10), at some distance x_0 above A, which may be assumed the same for each post. If these points are

known then the reactions, H and V, may be applied at I and I' and the analysis proceeded with as before. The distance c' is to be substituted for c in all the preceding equations. The effect of this modification will be to reduce greatly all the portal stresses. With a well-braced portal the point of inflection may be taken, approximately, as midway between the lower edge of the portal and the base of the post. The value of c' is then $\frac{1}{2}(c - e) + e$ and the moments at D and D' will be one-half of the values given by eq. (3). There will also be a bending moment at the foot of each post equal to $H \times \frac{c - e}{2} = \frac{P(c - e)}{4}$, the same as at D and D'.

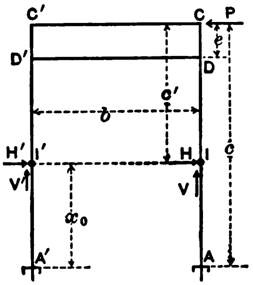


FIG. 10.

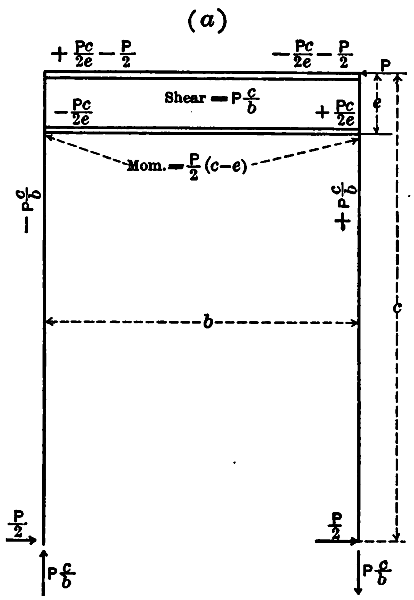
If the maximum resisting moment of the end post, calculated as above explained, is less than $\frac{P(c - e)}{4}$, then the end is not fixed, but the bending moment developed there will be equal to the resisting moment. Call this moment M. The point of inflection may then be found by writing $\frac{P}{2} x_0 = M$, where x_0 is the distance of this point above A. Where end floor beams are well connected to the end posts the latter may be considered as fixed in all cases.

The preceding discussion applies also to the sway-bracing of elevated railroad structures where the columns are fixed at top and bottom. For a more exact solution, where the upper ends of the posts are not rigidly fixed, see Art. 199.

187. *Summary of Portal Stresses.*—For convenience in designing, the maximum stresses in the various forms of portals discussed are summarized in Fig. 11. The distance c is to be taken as the actual length of the end post if the posts are assumed as hinged at the base. If they are assumed as fixed, it is the distance to the point of inflection. The stresses in the posts are given on form (a) only.

188. **Effect of Portal Stresses on the Vertical Trusses.**—The wind stresses have now been followed as far as A and A', the reactions H, H', V, and V' having been determined. These forces are in the plane

of the portal. Fig. 12 shows a side view of the joint at A where the end post is inclined. The force V is the tension in $A\,C$ as already found. To resist this force requires a downward reaction R equal to



$V \cos \theta$, and a compressive stress S in the lower chord equal to $V \sin \theta$. On the opposite side of the truss an equal upward reaction exists and an equal tension S in the lower chord.

Fig. 13 shows the lower chord system of the truss and the forces S acting at A and A'. If the truss is symmetrical there will be equal forces S' acting at the other end. This system of forces causes a uniform compression in the chord $A\,B$ and a uniform tension in the chord $A'\,B'$ equal to S. If the portal at the right end is

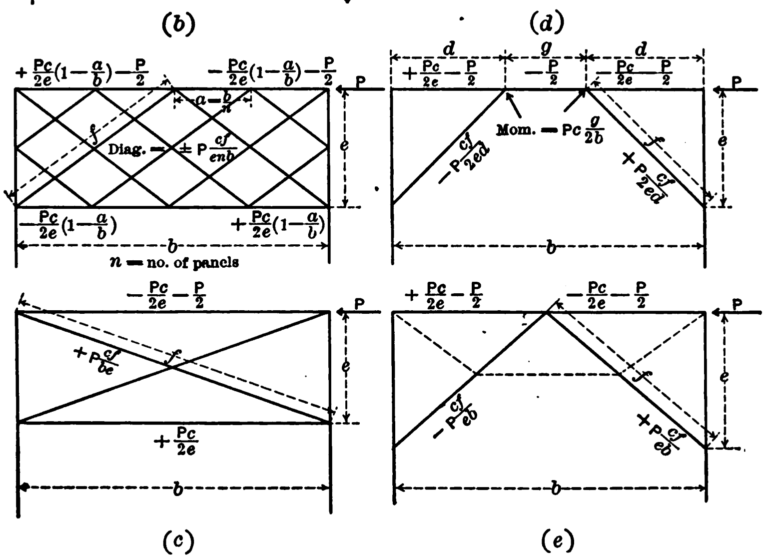


FIG. 11.

$P_1 E = V \tan \phi$

vertical the forces S' disappear (Fig. 14), and the forces S are balanced by lateral reactions R', which give rise to small stresses in the lower lateral system and a stress in the lower chord varying from S at the left end to zero at the right.

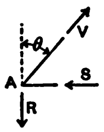


FIG. 12.

No stresses are caused in the upper chord by the portal truss, as the forces acting at C and C' which have already been considered are in complete equilibrium.

189. Skew-Portals.—Fig. 15 is a plan of the upper lateral system of a skew-bridge with vertical end posts, and Fig. 15 (a) is the vertical projection of the portal upon a parallel plane. The lateral force, P, applied at C, is resolved into the components $P \sec \beta$ and $P \tan \beta$, where β = angle of skew. The force $P \sec \beta$ replaces the force P in the discussion of the preceding article, and $P \tan \beta$, together with a similar force at the right end, acts upon the main truss, producing small additional stresses.

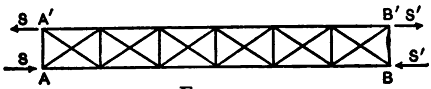


FIG. 13.

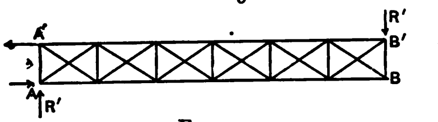


FIG. 14.

Fig. 16 represents the upper laterals and the portal of a skew-bridge having inclined end posts. The lower laterals are shown by dotted lines. Fig. (b) shows the portal projected on a plane parallel to itself. The component of the force P applied in the plane of the portal is $P \sec \beta$ as before. The distance $c = \sqrt{\overline{AE}^2 + h^2}$ and $AE = AF \cos \beta$. In the analysis of portal stresses the reactions are conveniently resolved into the components H and V, in which V is parallel to the end post. The moment in the end post at D is then $H(c - e)$, and the direct stress is V. The moments in the portal will be found

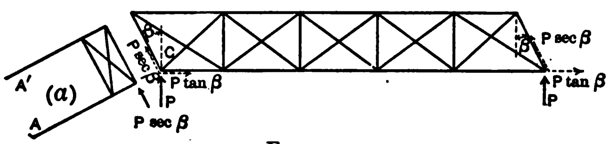


FIG. 15.

as before; the shear at right angles to the flange will be $V \cos \phi$. The stress in the lower chord developed at A will be $V \sin \theta + H \sin \beta$.

190. The Overturning Effect of Wind Pressure on Trains.—Fig. 17 represents the forces acting at a panel point. The panel wind load on the train is P. The resistance to this force is supplied by the lateral

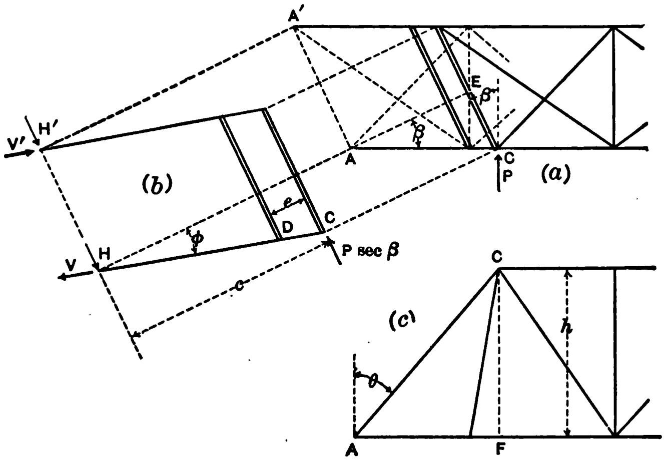


FIG. 16.

system in the plane $A\,A'$ at a distance h below the centre of wind pressure. The reaction H' is equal to P, but there are also vertical reactions at A and A', each equal to $P\frac{h}{b}$. These are supplied by the vertical main trusses; that is, the wind pressure produces a downward load at each panel point of the leeward truss equal to $P\frac{h}{b}$, and an equal upward load on the windward truss. The stresses due to these loads must be calculated and combined with the other wind stresses. Those in the windward truss will all be equal to, but of opposite sign from, those in the leeward truss. The stresses in the web members are small and may usually be neglected; those in the lower chords should be determined.

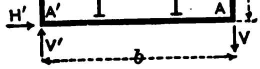


FIG. 17.

On the leeward side the lower chord stresses are therefore increased by wind pressure in three ways: (a) as the tension chord of the lower laterals, (b) a tension from the portal equal to $V \sin \theta$, and

(*c*) a tension from the vertical truss due to the overturning effect. The total wind stress thus becomes a large percentage of the live- and dead-load stresses. On the windward side all these wind stresses are compressive and the total may easily exceed the tensile stresses from vertical loads. The most favorable condition for this reversal is when the bridge is loaded with a train load of minimum weight.

191. **Transverse or Sway-Bracing.**—Transverse bracing of the same form as portal bracing is usually placed at each panel point of a deck-bridge and at each panel point of a through-bridge when the height is more than about 25 feet. This bracing is sometimes designed to carry the wind pressure from one chord to the other at each panel point, in which case but one lateral system is needed. The stresses are computed in the same way as in portal bracing, the external lateral force being equal to the wind load upon one panel. The resulting vertical reactions corresponding to V and V', Art. 185, act as loads, upward or downward, upon the main trusses. The portal bracing itself is now subjected to the same loads as the intermediate sway-bracing.

When two lateral systems are used, the stresses in the sway bracing are still often computed on the same assumptions as the above, although if the lateral systems have equal lateral deflections these stresses are zero. However, with wind pressure upon the *unloaded* bridge, the lateral system of the chord supporting the floor has, in the case of railway bridges, only about one-third of its full load, while the other system is fully loaded. In this case the lateral deflections are not equal; the sway-bracing will be distorted, and some stress will be thereby transmitted to the stiffer lateral system. The assumption that one-half the wind pressure upon the one system is thus transferred is on the safe side when the portals are properly designed, even with the most rigid form of sway-bracing; with a flexible form, as simple knee-braces, very little stress can come upon this bracing from wind pressure.

The chief purpose of the sway-bracing is to stiffen the structure against lateral vibrations. It is generally constructed of minimum practicable sections and made as deep as the head-room will permit.

Sometimes transverse bracing is designed to equalize the effect of eccentric loads, such as a load on one track of a double-track structure, so that the two main trusses will be equally deflected and hence equally loaded. Inasmuch as the sway-bracing depends upon the upper and

lower lateral trusses for its support in resisting such unequal deflections, and as the lateral trusses are much more flexible than the vertical trusses, such equalization of load cannot be accomplished to any appreciable extent. An exact analysis of this problem requires a consideration of the flexibility of all the different trusses and will not be discussed further at this point. This problem is considered with others of a similar nature in Part II.

192. End Bracing for Deck-bridges.—Fig. 18 (*a*) shows the bridge supported at the bottom and Fig. (*b*) at the top. The bracing may be placed in an inclined plane, as in the usual truss-bridge, or may be in

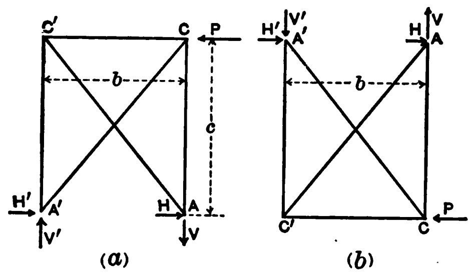


FIG. 18.

a vertical plane, as in the ordinary deck-plate girder. The diagonals are calculated as tension members. None but direct stresses exist, all of which are readily found by the usual methods.

There is a compression produced in $A'C'$, in both types, equal to $P\dfrac{c}{b}$. If the bracing is placed in an inclined plane the effect upon the chord stresses due to the direct stresses in the end posts is determined as in Art. 188. In type (*a*) the lateral force P is large, as it results from the pressure upon the loaded chord. The stresses in the end bracing and in the chords of the main truss are considerable. The overturning effect is calculated as in Art. 190, the overturning moment being calculated with reference to the plane of the *upper* laterals.

193. Stresses in Bridges on Curves.—These differ from the stresses in straight bridges from two causes: first, the centrifugal force from rapidly moving trains; and, second, the curvature of the track whereby the vertical load is no longer applied along the axis of the bridge. The effect of curvature will therefore be considered in two parts.

194. *Amount of the Centrifugal Force.*—The centrifugal force, F, of a body of weight, P, moving in a curve of radius r, is equal to

$$F = \frac{v^2}{gr} P, \qquad (18)$$

in which g = acceleration of gravity = 32.2 ft. per sec. per sec. If g be expressed in miles per hour and the curvature expressed in terms of the degree of curve, then $g = 32.2 \times 60^2/5{,}280 = 79{,}100$ and $r = \frac{5{,}730}{D \times 5{,}280} = 1.085/D$. Hence

$$F = \frac{v^2 D}{79{,}100 \times 1.085} \cdot P = .0000117\, v^2 D P. \qquad (19)$$

For any given speed and curvature we may write

$$F = kP. \qquad (20)$$

in which k is a constant and F and P are in like units. The velocity to be assumed in eq. (19) depends upon the degree of curvature; it is generally taken at what is considered to be the maximum safe speed.

In the specifications of the Maintenance of Way Association the centrifugal force is based on an assumed speed equal to $60-2\frac{1}{2} D$. According to this rule the values of k in eq. (20) for different speeds are as follows:

D	v Miles per Hour	k
1°	57½	0.039
2°	55	0.071
3°	52½	0.097
4°	50	0.117
5°	47½	0.132
6°	45	0.142
7°	42½	0.148
8°	40	0.149

For all curvatures sharper than 8° it is recommended that the centrifugal force be taken as for 8° or 15% of the vertical load.

195. *Stresses due to Centrifugal Force.*—We will now consider the effect of this centrifugal force upon the laterals, the main trusses, and the floor system.

The centrifugal force F is always proportional to the weight P,

being equal to $k P$, hence the horizontal pressure exerted upon a bridge by a train moving in a curve will at all points be equal to the vertical pressure or weight, multiplied by the constant k. If the moving load consists of a series of concentrated loads, then the horizontal forces will be a similar series of concentrated loads, each equal to k times the corresponding vertical load. This principle can be used to advantage in calculating stresses.

The general effect of centrifgual force is exactly the same as that of any other lateral force, such as wind pressure, acting on the train; it stresses the laterals belonging to the loaded chord and it causes an overturning effect, as shown in Art. 190. The line of action of the centrifugal force is along the centre of gravity of the load and is usually taken as 5 feet above the base of rail.

Fig. 19 represents the forces acting at a panel point. P is the vertical load and F is the centrifugal force applied at a distance h above the lower laterals as in Fig. 17. The load P is eccentrically applied by reason of the curved track and its inclination. In the following analysis these forces will be assumed as applied at the centre of gravity of the moving train as this is the true condition. However, if the inclination of the track accords with the speed, the resultant of F and P will pass through the centre of the track and hence these components may be considered as applied at that point. This assumption is often made, but it does not meet the case of a stationary load, where $F = 0$, and is not sufficiently general in its application to be very satisfactory.

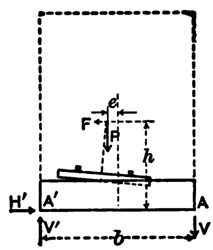


Fig. 19.

From the above considerations we may state the following methods of analysis:

Lateral Truss.—Calculate the stresses as for a vertical truss, for a series of loads each equal to k times the given vertical loads for *both* rails. The moments and shears will be equal to $2\,k$ times those found in the usual way for the vertical main trusses.

Vertical Trusses.—From Art. 190 the joint load on the outer truss due to a horizontal force F is equal to $F\dfrac{h}{b}$, where h = distance of

line of application of the centrifugal force above the plane of the lower laterals, b = distance centre to centre of trusses, and F = horizontal joint load due to centrifugal force. But $F = k$ times the corresponding vertical load P on one floor beam (both rails), hence the joint load on the outer truss due to centrifugal force $= k\frac{h}{b}$ times the total vertical load on the floor beam, or $2\,k\frac{h}{b}$ times the load on one rail. Then, since all joint loads bear this same ratio to the vertical loads, the stresses in all the members of the outer truss due to centrifugal force may be obtained by multiplying the live-load stresses, calculated in the usual way, by the constant $2\,k\frac{h}{b}$. The inner truss will receive stresses of opposite sign from those in the outer truss. They are of no significance except as contributing to the reversal of the lower chord stress.

Stringers.—Since the stringers constitute a simple bridge of a span one panel in length, the stresses in the laterals and in the stringers themselves due to overturning may be found as above described. A lateral system for the stringers should always be provided in bridges on curves; but, if it is not, each stringer may be assumed to carry one-half the lateral moment and shear due to the centrifugal force.

Floor Beams.—These will be stressed by reason of the overturning effect. The half of the beam adjacent to the outer truss will receive stresses equal to $2\,k\frac{h}{b}$ times those calculated for vertical load.

196. *Stresses Due to Eccentricity.*—The most convenient method of calculating the stresses due to a vertical load placed on a curved track is to calculate, first, the stresses for a centrally applied load in the usual way and then correct these stresses for eccentricity. The stresses due to centrifugal force require no correction as they are applied horizontally.

In Fig. 19 let P represent the total load on any floor beam, and let e be the average eccentricity for this load (= average eccentricity for a half-panel each side of this joint). This eccentricity refers to the centre of gravity of the load, and in determining its value the inclination of the track as well as its curvature must be considered. It will be assumed as positive when towards the outer truss. The joint loads

at A' and A due to P will be $\frac{P}{2} + P\frac{e}{b}$ and $\frac{P}{2} - P\frac{e}{b}$ respectively. Only the portions $+ P\frac{e}{b}$ and $- P\frac{e}{b}$ need be considered, as these are the amounts by which the loads vary from those produced by a central load. A plus sign represents a downward load and a minus sign an upward load, but as e may be plus or minus, we may have either kind of load on either side.

To calculate the exact effect of these loads, $P\frac{e}{b}$ and $- P\frac{e}{b}$, due to a series of concentrated loads, would be very difficult, as the value of e is different for different panels and the position of loads for maximum effect could not easily be determined. However, as the stresses under consideration are small and in the nature of a correction, it is sufficiently accurate to substitute an equivalent uniform load calculated as explained in Art. 172. The one equivalent load for moments at the quarter point may be used for all stresses. Having obtained this uniform load and determined the eccentricity of each panel point, then the load on each joint of the outer truss will be $P\frac{e}{b}$ and on the inner truss will be $-P\frac{e}{b}$, where P is the panel load for both rails for the equivalent uniform load. For chord stresses, assume all joints loaded; for web stresses load the same joints as usual in getting maximum stresses, irrespective of whether the loads be upwards or downwards. The resulting stresses are to be combined with those found by considering the track straight.

For the stringers the eccentricity may be taken as the average for the panel in question. This being constant for the panel, the stresses in the outer stringer may be found by multiplying those due to the actual wheel loads, centrally applied, by $\frac{2e}{b}$, and those on the inner stringer by $-\frac{2e}{b}$. For the floor beams the value of P should be taken as the maximum floor-beam load due to the actual wheel loads. The end reactions to be considered are then $P\frac{e}{b}$ and $- P\frac{e}{b}$. The maxi-

mum bending moments must be determined with reference to the spacing of the stringers, which may vary considerably at the different joints. The stress in the hip-vertical is equal to its maximum stress for vertical load, multiplied by $\frac{2e}{b}$.

If it is not desired to use an equivalent uniform load, the best method to employ is that of influence lines, getting the total stress from vertical load and from centrifugal force at one operation. Suppose in Fig. 20 (*a*) the line $x\,y$ represents the horizontal projection of the

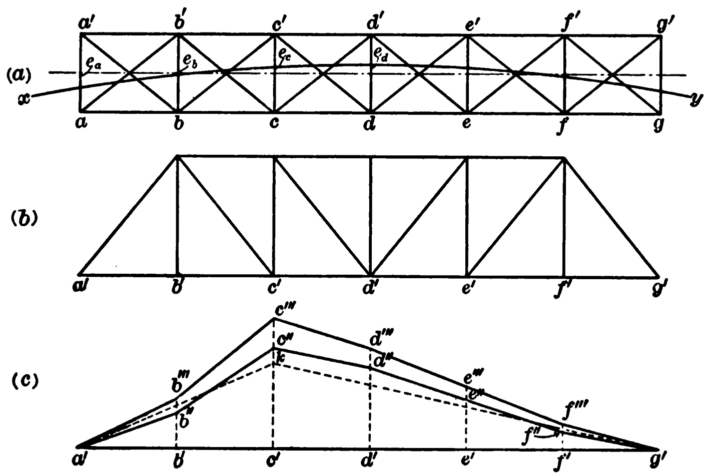


FIG. 20.

centre of gravity of the train (not the track centre). Let e_b, e_c, e_d, etc., be the eccentricities at the several panel points (the average for a half panel each side). Consider the bending moment at c', Fig. (*b*), for example; $a'\,k\,g'$, Fig. (*c*), is the influence line for a central load. To construct the influence line for an eccentric load, multiply the ordinate to the line $a'\,k\,g'$ at each panel point by the value of $\left(\frac{1}{2} + \frac{e}{b}\right) \times 2$ for the point in question. Then with these modified values construct the broken line $a'\,b''\,c''\,d''$, etc., which will be the true influence line for the total moment at c in the outer truss due to vertical load. For the inner truss the multiplier is $\left(\frac{1}{2} - \frac{e}{b}\right) \times 2$. The effect of cen-

trifugal force is now to be added to the line $a'' b'' c''$, etc., by adding ordinates at each point equal to the ordinates to $a' k g'$, multiplied by the constant $2k\frac{h}{b}$, giving the final influence line $a' b''' c'''$, etc. For the inner truss the centrifugal force is omitted. Calculations are then made graphically by trial. The same method may be used for web stresses and for stringers and floor beams. This method is especially applicable to skew-bridges on curves, a form of structure which frequently occurs.

EXAMPLE.—Let it be required to calculate the stresses in the outer truss of a 200-foot-span bridge located on a 6° curve, for Cooper's E–50 loading, assuming a speed of 40 miles per hour. Assume a panel length of 25 ft., a height of 32 ft., and a width of 20 ft. centre

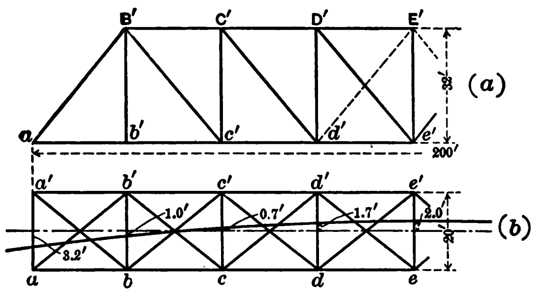


FIG. 21.

to centre of trusses. Fig. 21 (a) represents half of the outer truss and Fig. 21 (b) the lower laterals and line of the centre of gravity of the train. Assume the net eccentricities to be as shown in the figure.

We first calculate the stresses in the truss as in the ordinary case, by the usual wheel-load method.

The maximum moments and shears are as follows: Moments: at $b' = 6{,}784$; at $c' = 11{,}250$; at $d' = 14{,}010$; at $e' = 14{,}810$. Shears: $a' b' = 271.4$; $b' c' = 204.7$; $c' d' = 169.5$; $d' e' = 97.9$; $e' f' = 56.4$. The resulting stresses are given in the first lines of the tables below.

CHORD STRESSES.

Member.	$a' b'$	$b' c'$	$c' d'$	$d' e'$	$B' C'$	$C' D'$	$D' E'$
Vertical load, straight track	+ 212.0	+ 212.0	+ 351.5	+ 437.9	− 351.5	− 437.9	− 462.8
From laterals	0	+ 76.0	+ 126.0	+ 156.8	......	......	......
From overturning effect	+ 21.4	+ 21.4	+ 35.4	+ 44.2	− 35.4	− 44.2	− 46.7
From eccentricity	+ 14.1	+ 14.1	+ 34.0	+ 49.8	− 34.0	− 49.8	− 55.7
Total	+ 247.5	+ 323.5	+ 546.9	+ 688.7	− 420.9	− 531.9	− 565.2

WEB STRESSES.

Member.	$a' B'$	$B' c'$	$C' d'$	$D' e'$	$E' d'$	$B' b'$	$C' c'$	$D' d'$	$E' e'$
Vertical load, straight track	− 344.4	+ 259.9	+ 215.0	+ 124.1	+ 71.5	+ 94.5	− 169.5	− 97.9	− 56.4
From overturning effect	− 34.7	+ 26.2	+ 21.7	+ 12.5	+ 7.2	+ 9.5	− 17.0	− 9.8	− 5.7
From eccentricity	− 22.8	+ 31.2	+ 26.3	+ 16.2	+ 6.7	− 9.4	− 20.7	− 12.8	− 5.3
Total	− 401.9	+ 317.3	+ 263.0	+ 152.8	+ 85.4	+ 94.6	− 207.2	− 120.5	− 67.4

For a velocity of 40 miles per hour and a 6° curve the value of k of eq. (18) = .0000117 × 40 × 40 × 6 = .112. For the lower laterals the moments and shears due to centrifugal force will be equal to the moments and shears from vertical load, multiplied by $2\,k$ (= .224) In Fig. 21 (b) the diagonals which are in action are $a\,b'$, $b\,c'$, etc. The tension in $d'\,e'$ will therefore = moment at d divided by 20, that in $c'\,d'$ = moment at c ÷ 20, etc. We have then stress in $d'\,e' = \dfrac{14{,}010 \times .224}{20} = 156.8$; stress in $c'\,d' = \dfrac{11{,}250 \times .224}{20} = 126.0$, etc. These stresses are given in the second line of chord stresses.

For the overturning effect the distance of base of rail above laterals will be assumed at 4 feet, making the distance of the centre of gravity above the laterals about 9 feet, which is the value of h of Fig. 19. The stresses in all the members due to vertical loads must then be multiplied by $2 \times .112 \times \dfrac{9}{20} = .1008$. The stress in $a'\,b'$ is therefore equal to $+\,212.0 \times .1008 = +\,21.4$, etc. The results are given in the tables.

To calculate the corrections due to eccentricity we first get the equivalent uniform load for a 200-foot span. This is given in Fig. 47, Chapter V, and is equal to 3,000 lbs. per foot for one rail, or 6,000 lbs. for both rails. The panel load = 6 × 25 = 150. Then the loads on the outer truss are as follows: at $b' = 150 \times \dfrac{-1}{20} = -\,7.5$; at $c' = 150\,\dfrac{+.7}{20} = +\,5.25$; at $d' = 150\,\dfrac{+1.7}{20} = +\,12.75$, and at $e' = 150\,\dfrac{+2.0}{20} = +\,15$. The loads on the right are the same. Then with these loads applied on the truss at the several joints the chord stresses are to be calculated. The reaction = 18.0. The stresses are given in the table with correct signs. For web members the usual joints are to be loaded. The stress in the hip-vertical $= 94.5 \times \dfrac{2 \times (-1)}{20} = -\,9.4$, that is, a compressive stress of 9.4. This is the only member for which stresses of opposite sign appear in the table.

If the inner truss be analyzed, the stresses from the laterals would be compressive, that in $a\,b$ being 76.0, in $b\,c$ 126.0, etc. The stresses from overturning and from eccentricity would all be equal to but of opposite sign from those given in the table. The maximum stresses would occur for a zero velocity, in which case only eccentricity would be considered. One member only, the hip-vertical, would have a stress greater than the corresponding member of the outer truss.

197. Stresses Due to Tractive Forces.—When a train crosses a bridge with brakes set, the horizontal force exerted upon the track by the braked wheels may need to be considered. The amount of the force is generally taken at 20% of the vertical load. A portion of this

is transmitted through the track to the road-bed beyond the bridge, but this amount is uncertain and will be small under some conditions, so that it cannot be considered in the calculations. In the through-bridge, and in the deck-bridge supported at the level of the top chord, the only members stressed by this tractive force are the floor system and the lateral truss belonging thereto, including the chord members of the loaded chord. With the usual lateral diagonals (Fig. 22), connected to the stringers at d and c, the tractive force

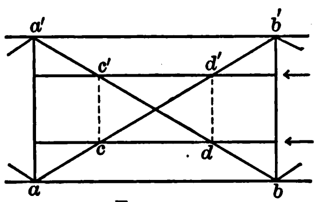


FIG. 22.

is resisted mainly by the bending resistance of the floor beams $a\ a'$, $b\ b'$, etc., the load on each being that for one panel length. The addition of members $d\ d'$ and $c\ c'$ forms a truss of the laterals and stringers which will then resist the tractive force by direct stresses unimportant in amount. The chord members receive a tensile stress when the train approaches from the anchored end and a compressive stress when it approaches from the expansion end. This stress increases uniformly from the roller to the fixed end, receiving an increment at each panel point.

In deck structures the tractive forces cause some modification of the vertical reactions and hence stresses throughout the structure. If T = total tractive force for the span (Fig. 23), then $R_1 = T\frac{h}{l}$. The horizontal reaction, H, will be applied at the fixed end. If, for example, T = 20% the total vertical load and $h = l/5$, then R_1 will equal about 4 per cent. of the reaction for full vertical load. This indicates roughly the relative importance of these stresses.

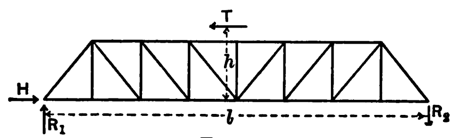


FIG. 23.

198. Elevated Railroad Bents.—Transversely, an elevated railroad bent is similar to the plate-girder portal already considered in Art. 185 (*a*). The posts are usually fixed at the base, giving a point of inflection midway between the base and the cross-girder. In a longitudinal direction several posts are generally rigidly connected by the longitudinal girders or trusses, expansion joints being introduced about

every third or fourth panel. The bracing of the bents longitudinally is often accomplished by full diagonal bracing between a pair of columns, thus forming a tower in each section of the structure. Where this cannot be done the bending resistance of the columns must be depended upon to resist the longitudinal horizontal forces in the same manner as they do the lateral forces. The construction thus becomes a continuous girder, fastened to vertical columns, which may be either free or fixed at the base.

Fig. 24 represents a 5-column section in which the columns are of equal length and cross-section. T = total tractive force acting on the

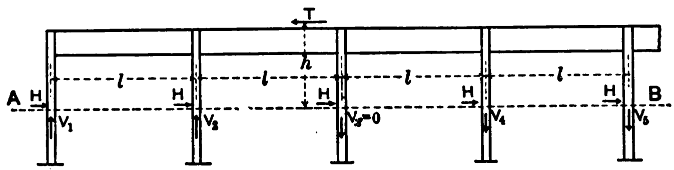


FIG. 24.

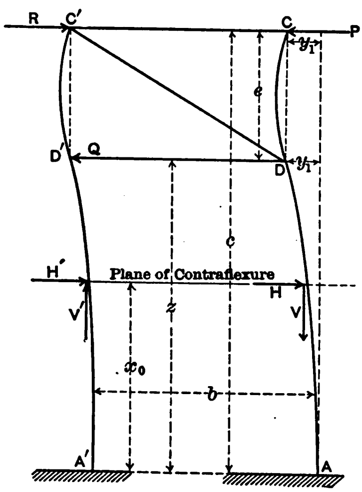
structure. Let $A\,B$ be a plane through the points of inflection of the columns (midway between girder and base in the case of fixed columns and at the base for columns hinged at the base). The columns being equally flexible the horizontal reaction at each column will be the same and equal to $1/5\,T$. The vertical reaction will not be important, but may be found by taking moments of T about the centre of gravity of the five columns considered as one structure. With equally spaced columns of equal cross-section the moment centre will be at the centre of the group. The vertical reaction at each column will be proportional to its distance from the moment centre and the moment of this reaction will be proportional to the square of the distance. Thus $V_5 = 2\,V_4$, $V_1 = 2\,V_2$ and $V_2 = V_4$. Hence

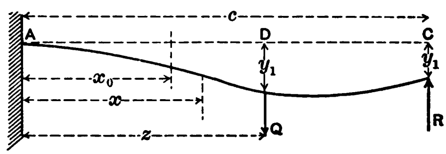
$$T\,h = V_1 \cdot 2\,l + V_2\,l + V_4\,l + V_5 \cdot 2\,l = 10\,V_2\,l; \text{ and } V_2 = T\frac{h}{10\,l}.$$

If the columns are of different cross-sections their deflections for a given load will be inversely proportional to their moments of inertia about an axis perpendicular to the plane of bending. Since these deflections must be equal it follows that the resistance, H, at the base, will be directly proportional to the moment of inertia of the column. Hence the total force, T, will be divided among the several columns in

proportion to their moments of inertia. For a more general treatment of this class of problems, see Part II.

199. Portal Frames with Posts fixed at the Base, but Upper Ends not fixed.—Let Fig. 25 represent a portal, or a bent of an elevated railroad or of a steel-frame building, having the posts fixed in direction at the ground and having a single system of diagonal bracing as shown. The problem is to find the point of inflection x_0 from the base, and then the values of the reactions H and V upon the columns. These reactions will be the same as in the portals of Art. 185,





(a)

FIG. 25.

when the point of inflection is treated as the base of the column. It will be assumed in this analysis that the longitudinal distortion of the members of the truss is small as compared to the bending of the posts, so that the former will be neglected. This is, in most cases, sufficiently accurate for all practical purposes. A more general treatment of the subject, taking into account all distortions, is given in Part II.

Under the assumption noted, the points D and C deflect equally. Fig. (*a*) represents the column $A\,C$ separated and the forces indicated. From the general differential equation of the elastic line, $E\,I\,\dfrac{d^2 y}{d\,x^2} = M$, we have for $x < z$,

$$E\,I\,\frac{d^2 y}{d\,x^2} = R\,(c - x) - Q\,(z - x). \quad . \quad . \quad . \quad (21)$$

Integrating eq. (21) between the limits o and z, we have

$$E\,I\frac{d\,y}{d\,x} = R\int_0^z (c-x)\,d\,x - Q\int_0^z (z-x)\,d\,x = R\left(c\,z - \frac{z^2}{2}\right) - Q\frac{z^2}{2}, \quad (22)$$

which is $E\,I$ times the angle of the deflected column at D.

For $x > z$,

$$E I \frac{d^2 y}{d x^2} = R (c - x),$$

from which

$$E I \frac{d y}{d x} = R \left(c x - \frac{x^2}{2}\right) + C, \quad . \quad . \quad . \quad . \quad (23)$$

in which C is the value of $E I \frac{d y}{d x}$ at D, as given by (22).

Hence for $x > z$

$$E I \frac{d y}{d x} = R \left(c x - \frac{x^2}{2}\right) + R \left(c z - \frac{z^2}{2}\right) - Q \frac{z^2}{2}. \quad . \quad (24)$$

Integrating this again from z to c, we obtain the deflection at C as compared to that at D. But by the conditions of the problem, D and C deflect equally, and hence this last integral is zero, or from (24),

$$E I y = R \int_z^c \left(c x - \frac{x^2}{2}\right) d x - Q \int_z^c \frac{z^2}{2} d x =$$

$$R \left(\frac{c^3}{3} - \frac{c z^2}{2} + \frac{z^3}{6}\right) + Q \left(\frac{z^3}{2} - \frac{z^2 c}{2}\right) = 0. \quad . \quad . \quad . \quad (25)$$

whence

$$\frac{R}{Q} = \frac{3 z^2}{2 c^2 + 2 c z - z^2}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For the point of inflection we have $R (c - x_0) = Q (z - x_0)$ or

$$\frac{R}{Q} = \frac{z - x_0}{c - x_0}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

whence from (26) and (27)

$$x_0 = \frac{z}{2} \left(\frac{z + 2 c}{2 z + c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This gives the point of contraflexure at which H and V (as in Art. 185) are to be applied. The remaining analysis is exactly the same as there given, using here $c - x_0$ in place of c.

The requisite strength of anchorage may be computed from the bending moment at the base of the column, $= H x_0$.

This problem is usually solved by assuming the point of inflection ½ z from the base. For the extreme case where $e = z = \frac{c}{2}$, eq. (28)

gives $x_0 = \frac{5}{8} z$. When e is less than this, the point of inflection approaches the middle point between A and D, so that it may be said this point lies somewhere between $\frac{1}{2} z$ and $\frac{5}{8} z$ above the base for all cases of fixed base. But this base is never perfectly fixed in direction, and any flexibility here would lower the point of inflection. Neither is the web bracing above perfectly rigid, and any distortion here would raise the point of inflection, so that these assumptions may be considered as offsetting each other, and the formulas applied rigidly as above.

200. Framed Bents in Buildings.—The framed bent in a building, such as shown in Fig. 26, is in the nature of a portal. The columns

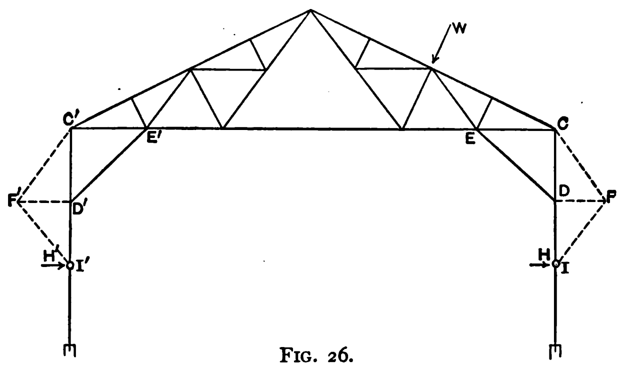


FIG. 26.

may be fixed or free at their bases. In the former case the points of inflection, I and I', are closely determined by the analysis of Art. 199. Knowing these points the analysis for wind pressure is simple. The horizontal components H and H' are assumed equal. The stress in $E D$ is found at once, and the shear and compression in the post at C. A similar analysis on the left gives the forces acting at C' and E'. All the external forces acting on the roof truss are then known and its analysis may be proceeded with.

Graphically, the analysis of the roof-truss stresses can be proceeded with most readily by substituting for the vertical beam, $I C$, a truss, $I C F$, the length $D F$ being any arbitrary length. The members of this truss will receive direct stress only and the analysis by means of the force polygon can therefore be begun at I and carried through in the same manner as for any truss. The resulting stresses will be

correct for all members except the posts; these are best analyzed directly as already explained.

Example.—Let it be required to analyze the truss shown in Fig. 27 (*a*) for a wind pressure of 30 lbs. per sq. ft. on a vertical surface. The pressure normal to the roof is 22 lbs. Spacing of trusses = 15 ft. The apex loads are as given in the figure.

Fig. (*b*) gives the analysis. The load-line is *A B–F–K*. The

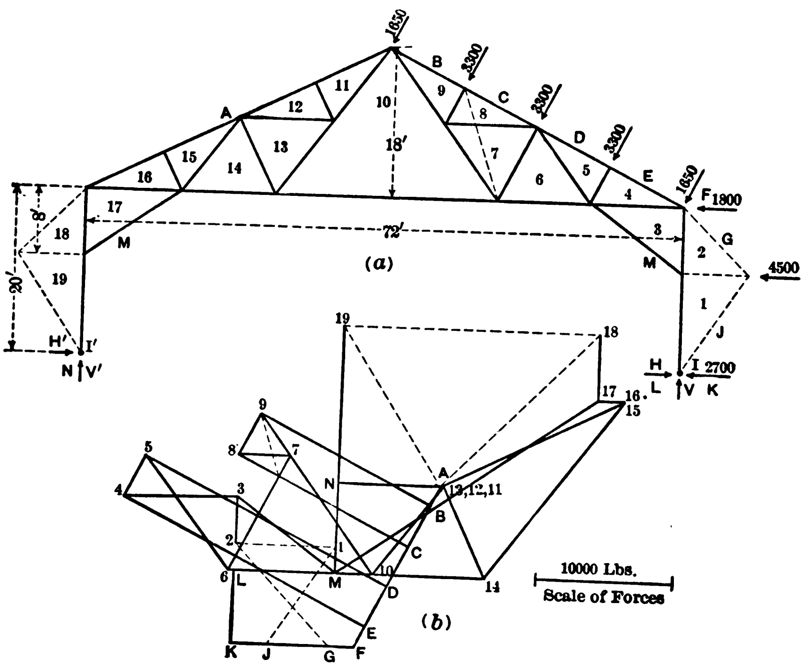


FIG. 27.

horizontal components of the reactions, *H* and *H′*, are assumed as equal and the vertical component, *V′*, is conveniently found analytically by taking moments about *I*. The value of *V* is then found by closing the polygon by drawing *K L M N A*. Beginning at *I* the stress diagram is drawn as explained by several examples in Chapter II.

201. Stresses in Trestle Towers.—*Stresses in a Single Vertical Bent.*—Fig. 28 represents the usual form of vertical bent. The bracing

consists of horizontal members $A\,B$, $C\,D$, $E\,F$, etc., capable of resisting either tension or compression, and diagonal braces which are generally designed for tension only. Even if made of angles or other shapes their length is usually so great as to make them ineffective as compression members. They will therefore be assumed to act in tension only. The posts $A\,E$ and $B\,F$ are assumed to have the same inclination or batter, but they are sometimes made of different inclinations in the case of trestles on curves. It will be convenient to consider separately the stresses due to: (*a*) the vertical loads and (*b*) the horizontal forces such as wind load and centrifugal force.

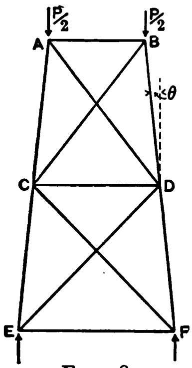


FIG. 28.

(*a*) *Stresses Due to Vertical Loads.*—For symmetrical loading, the loads brought to points A and B, due to live and dead weight above this level, are equal, and by reason of symmetry the diagonal bracing will receive no stress from such loading. Therefore, stress in $B\,D$ and $A\,C = \frac{1}{2}P \sec\theta$ and stress in $A\,B = \frac{1}{2}P \tan\theta$. Below D the additional dead load applied at D must be considered, and so on.

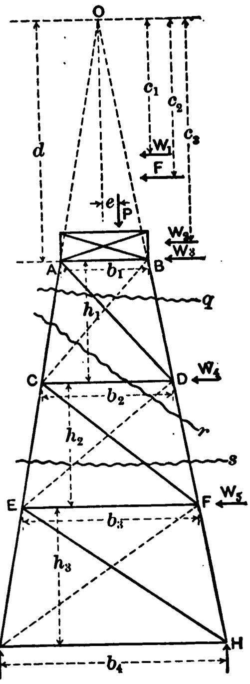


FIG. 29.

If the loads are unsymmetrically applied, as in the case of a double-track structure, or a structure on a curved track, the loads at A and B will not be equal. If the load at B is the greater then the diagonals $A\,D$, $C\,F$, etc., will be brought into action; the other diagonals will be relieved.

In Fig. 29, e is the eccentricity of the load with reference to the centre line, and O is the intersection of the centre line of the posts at a distance d above the plane $A\,B$. The stresses in the various members may readily be obtained graphically by determining the loads at A and B and constructing the force diagram as usual. Analytically, the stresses can best

be found by moment equations. With the notation as shown, the stresses are as follows:

Section q, moment centre at D,

$$AC = \frac{P(\frac{1}{2}b_2 - e)}{b_2} \cdot \frac{\overline{AC}}{h_1}.$$

Moment at A,

$$BD = P\frac{\frac{1}{2}b_1 + e}{b_1} \cdot \frac{\overline{BD}}{h_1}.$$

Moment at O,

$$AD = P \cdot \frac{e}{d + h_1} \cdot \frac{\overline{AD}}{b_1}.$$

Section r, moment at O,

$$CD = P\frac{e}{d + h_1}.$$

Section s, moment at F,

$$CE = P\frac{\frac{1}{2}b_3 - e}{b_3} \cdot \frac{\overline{CE}}{h_2}.$$

Moment at C,

$$DF = P\frac{\frac{1}{2}b_2 + e}{b_2} \cdot \frac{\overline{DF}}{h_2}.$$

Moment at O,

$$CF = P\frac{e}{d + h_1 + h_2} \cdot \frac{\overline{CF}}{b_2},$$

etc., etc.

This method applies equally well to bents where the posts have unlike batters, the eccentricity being measured from the intersection point O.

(*b*) *Stresses Due to Horizontal Forces.*—These include the wind pressure on the train and on the girder resting upon the bents, the pressure upon the tower itself, assumed as concentrated at the joints, and the centrifugal force acting at the centre of gravity of the train. The simplest manner to treat these horizontal forces analytically is to consider them applied at the proper level as indicated in Fig. 29. F is the centrifugal force acting on one bent; W_1 = wind pressure on train; W_2 = wind pressure on bridge or girder; and W_3, W_4, W_5, etc., are the wind loads on the tower itself. The values of F, W_1 and W_2 are calculated for half of each of the two spans supported by the bent in

question. The distances c_1, c_2 and c_3 are first calculated. Then the stresses are found as before by simple moment equations. For example, with section q, and moment at D, we have

$$A\,C = \frac{W_1(d+h_1-c_1)+F(d+h_1-c_2)+W_2(d+h_1-c_3)+W_3h_1}{b_2} \cdot \frac{\overline{A\,C}}{h_1}.$$

Graphically, the stresses may be found by a force diagram, first getting the forces (vertical and horizontal), acting at A and B, equivalent to the forces W_1, F and W_2: or, more simply, by getting the resultant P (Fig. 30), of these forces and considering it applied at the apex of an extension, $A\,O'\,B$, of the bent. The diagram may then be drawn in the usual manner. The stress in $A\,B$ will depend upon the assumption as to the manner of resisting the pressure upon the train and girder. To assume it resisted equally at the two points, A and B, is sufficiently accurate. This is taken account of in the analysis by making $A\,O'$ and $B\,O'$ equally inclined; the resulting stress in $A\,B$ will then be correct.

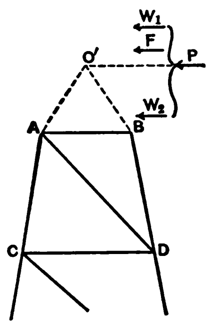


FIG. 30.

In the case of a double-track structure the maximum stress in the bracing will occur when the track to the windward is loaded and the wind pressure is a maximum.

(*c*) *Stresses Due to Traction.*—In a high trestle the stresses due to traction may be very considerable. If T = total tractive force per tower, the horizontal shears at all sections will equal T, giving the horizontal components in the bracing; the stresses in the posts will be found by moments. If the lengths of members as shown in the vertical projection of the tower be used in the calculations, then the resulting stresses will be the vertical components of the post stresses.

202. *Inclined Bents.*—If the bent itself is inclined, thus forming a tower in which the planes of all sides are inclined, it may conveniently be analyzed by considering, first, its projection on a transverse plane as a vertical bent, and finding its stresses as described in the preceding Article. The true stresses in all the members are then equal to the stresses so determined, increased by the ratio of the true lengths of the members to their projected lengths.

203. *Stresses at the Pedestals.*—If the resultant of the stresses in $E\,H$ and $F\,H$, Fig. 29, is a tension, then anchorage is required at H. The stress in $G\,H$ will depend upon where the fixed point of the trestle tower is located. If at G, then $G\,H$ must carry all the lateral force due to $E\,H$ and $F\,H$, less the frictional resistance at H. If H is the fixed point, $G\,H$ becomes stressed when the member $G\,F$ acts. This member must also be sufficiently strong to overcome the friction at the expansion point under dead load.

204. Symmetrical Towers Polygonal in Plan.—Towers for supporting fixed loads, such as water tanks, are generally made square or polygonal in plan. Fig. 31 represents such a tower with six posts and the usual diagonal and lateral bracing. The diagonals resist tension only.

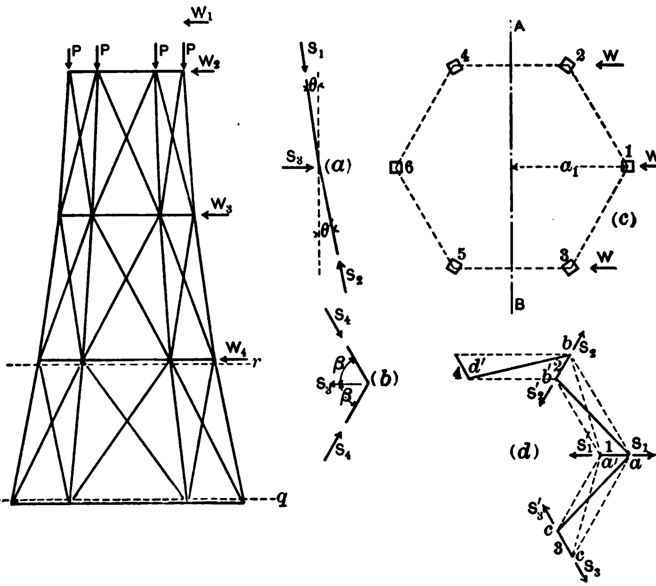


FIG. 31.

205. *Stresses due to Vertical Loads.*—For vertical loads the diagonals are not stressed. If θ = angle of inclination of the posts at any bent and P = total vertical load per post above any given story, then the stress in each post = $P \sec \theta$ and the horizontal component is $P \tan \theta$. At the top of the tower the latter component must be resisted by the top framework or other structure; and if the posts have different

inclinations in successive stories the lateral struts will be stressed at each level. In Fig. (*a*) S_1 and S_2 are the stresses in two successive posts and θ and θ' their angles with the vertical. The necessary horizontal reaction, S_3, in the plane of the posts, $= S_2 \sin\theta' - S_1 \sin\theta$ $= P (\tan\theta' - \tan\theta)$; and in Fig. (*b*) the resulting stresses in the struts of the polygonal frame will be $S_4 = \frac{1}{2} S_3 \sec^* \beta$.

206. *Stresses due to Wind Pressure.*—The stresses due to wind pressure may be found as follows: Consider the lowest story. Imagine a section q passed through the posts just above the lower struts, but consider the diagonal rods still attached to the posts. Fig. (*c*) represents this section. Now the tower as a whole may be considered as a vertical cantilever beam acted upon by lateral forces W_1, W_2, etc. The bending moment at section q is equal to the sum of the moments of all of these forces with respect to the plane of this section. Call this M. Determine the moment of inertia of the posts in Fig. (*c*) with reference to a gravity line $A\,B$ at right angles to the wind pressure; for this purpose each post area may be called unity. If $a =$ distance of any post from $A\,B$, then $I = \Sigma\, a^2$. The vertical component of the stress in any post, distant a from $A\,B$, may then be determined by the usual formula for flexure and is

$$S_v = M \frac{a}{\Sigma\, a^2}. \qquad \qquad (29)$$

The vertical components of the stresses in all the posts are thus found.

In the same manner get the vertical components of the stresses at the top of this same story, assuming a section r taken just below the lateral struts but above the diagonals. (This might also be taken above the struts as the vertical components are not affected by these horizontal struts.) In this way the stress at each end of each post, beyond the point of attachment of the diagonals, may be determined.

The stresses in the diagonals must now be found. Fig. (*d*) represents a horizontal plan of four of the posts together with the bracing. The stresses S at each end are known. From symmetry, the stress in the bracing between posts 1 and 2 is the same as that between 1 and 3, wind acting as shown; and it is also evident that S_1 is greater than S'_1. These conditions require the diagonals $a\,b'$ and $a\,c'$ to be in action and to have equal stresses. From equilibrium at a we then have

$$V.\,\text{comp.}\ a\ b' = V.\,\text{comp.}\ a\ c' = \tfrac{1}{2}\,(V.\,\text{comp.}\ S_1 - V.\,\text{comp.}\ S'_1). \quad (30)$$

The stress in $a\ b'$ being known we can pass to the post $b\ b'$ and determine the stress in $b\ d'$ by writing $\Sigma\ V = O$ for the stresses S_2, S'_2, $a\ b'$, and $b\ d'$; and so on, continuing around the tower. The same method applied to each story will determine the stresses in all the diagonals. The stresses in the lateral struts, $a\ b$, $a\ c$, etc., are then readily found by the equilibrium of all the forces at a joint, projected upon a horizontal plane.

The direction of wind pressure producing maximum stresses will be that of the longer axis of the polygon, for the moment of inertia of the section is the same about all axes, and hence from eq. (29) the post stresses will be a maximum when the distance a is a maximum.

CHAPTER VII

DEFLECTION OF STRUCTURES AND STRESSES IN REDUNDANT MEMBERS

Section I.—Deflection of Structures

207. The Displacement of any Joint of a Framed Structure Resulting from any Small Change of Length of any Member.—When for any reason the length of any member of a framed structure is changed, as by stress or a change in temperature, there will result in general a slight shifting in position of all parts of the structure (excepting those points held rigidly to the supports), to a new position corresponding to the new length of the affected member. If the lengths of several members are changed, the influence on the shape of the truss will be the combined result of all the individual changes. If these changes in length are small, it will be sufficiently exact to assume that the total displacement of any joint can be calculated by finding the displacement due to the change in length of each member separately and combining the results. That is to say, the effect of the change in any one member is practically independent of the effects produced by the others. Expressed in mathematical terms, the shape of the truss and position of the joints are functions of the lengths of all the members, and if an increment is added to any or all the members, the total effect upon the position of any joint can be had by differentiating the function with respect to each variable member separately and adding the results. Two methods of procedure will now be explained.

Let $A\ B$, Fig. 1, be any framed structure, fixed in position at A, O any joint, and $C\ D$ any member. Suppose it is required to determine the movement of O in any given direction $O\ Q$, due to a small change of length in $C\ D$, and also the total movement due to changes of length in any number of members.

Let l = length of $C\,D$;
z = given small change of length in $C\,D$;
d = movement of O in the direction $O\,Q$ due to the change of length z;
$D = \Sigma\, d$ = total movement of O due to the changes of length in any number of members.

208. *First Method of Calculation.*—It is evidently possible to express the position of O, with reference to a fixed plane $N\,N$ at right angles to $O\,Q$, in terms of the length of $C\,D$ and other fixed dimensions of the truss. This being done, a differentiation of this expression with respect

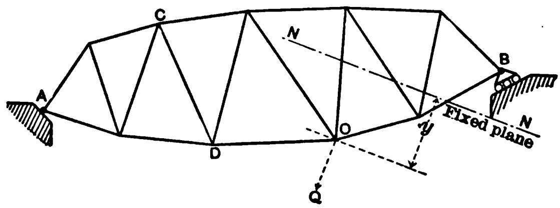


FIG. 1.

to the length l will give the desired relation between d and z, since these small quantities will have practically the same ratio as the differentials.

We therefore have in general

$$y = \text{function of } l = f(l) \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{d}{z} = \frac{d\,y}{d\,l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence

$$d = z\frac{d\,y}{d\,l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For a change of length in several members we may write

$$D = \Sigma\, d = \Sigma\, z\frac{d\,y}{d\,l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which the product $z\dfrac{d\,y}{d\,l}$ is to be calculated separately for each member.

To illustrate this method take the simple frame $A B C$, Fig. 2, and determine the downward movement of B due to a change of length z_1 in $A B$, and z_2 in $B C$. The position of B with reference to a fixed horizontal plane $A D$, in terms of the lengths of $A B$ and $B C$, is given by the expression

$$y = \sqrt{l_1^2 - l_2^2}. \quad \ldots \ldots \quad (5)$$

Consider first a change in l_1. Differentiating (5) with respect to l_1, we have

$$\frac{dy}{dl_1} = \frac{l_1}{\sqrt{l_1^2 - l_2^2}} = \frac{l_1}{y}.$$

Hence for a change in length of z_1 in l_1 we have

$$\frac{d_1}{z_1} = \frac{dy}{dl_1} = \frac{l_1}{y},$$

or

$$d_1 = z_1 \frac{l_1}{y}. \quad \ldots \ldots \quad (6)$$

In the same manner we find, for a change of z_2 in l_2

$$d_2 = -z_2 \frac{l_2}{y}. \quad \ldots \ldots \quad (7)$$

The total movement is

$$D = \Sigma d = z_1 \frac{l_1}{y} - z_2 \frac{l_2}{y}. \quad \ldots \ldots \quad (8)$$

From (8) the movement of B can readily be calculated for any small changes z_1 and z_2. It will be noticed in (6) and (7) that in each case the value of d is equal to the quantity z, multiplied by the differential coefficient of y with respect to l.

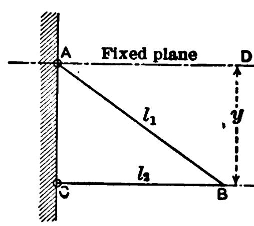


FIG. 2.

The method of calculation here given is simple in theory but is not readily applied to structures with numerous members. A much simpler method, and one especially adapted to use by those familiar with stress calculations, will now be given.

209. *Second Method of Calculation.*—Consider again the structure of Fig. 1. Suppose a force W to be applied at O and acting in the direction $O Q$ (Fig. 3). This force will cause stresses and deformations

in the various members of the structure and a deflection of the structure in the direction $O\ Q$. Each member stressed will contribute towards this deflection and among others the member $C\ D$ will be slightly elongated or compressed and will contribute a certain increment toward this movement. If this increment be determined in any manner, the ratio of such increment to the distortion in $C\ D$ will be only another expression for the differential of y with respect to the length of $C\ D$, since it is immaterial what causes the change in length of $C\ D$, the effect on the position of O will be the same. It is now proposed to explain a

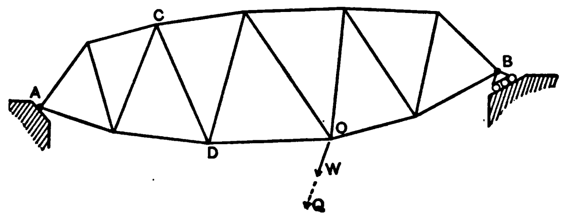


FIG. 3.

simple process of finding, in the case assumed, the movement of O due to the deformation of $C\ D$.

Let P = stress in member $C\ D$ due to load W,
λ = deformation of $C\ D$ due to stress P,
δ = movement of O due to deformation λ in $C\ D$.

During the movement of the point O the force W performs work, the amount of which is equal to the average force, $\frac{1}{2}W$, multiplied by the displacement δ, or work $= \frac{1}{2}W\delta$. This work is applied to the member $C\ D$ in causing the distortion λ. It can, therefore, also be expressed as the work done upon this member. As the average stress in the member is $\frac{1}{2}P$ the work is equal to $\frac{1}{2}P\lambda$. Hence we have

$$\tfrac{1}{2}W\delta = \tfrac{1}{2}P\lambda, \qquad (9)$$

or

$$\frac{\delta}{\lambda} = \frac{P}{W}. \qquad (10)$$

The value of δ/λ is, therefore, the desired ratio of movement of O to change of length in $C\ D$. This is seen to be equal to the ratio of stress P, to the load W applied at O, a quantity which is easily calculated.

We therefore have in general

$$\frac{dy}{dl} = \frac{\delta}{\lambda} = \frac{P}{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Returning now to the general problem of finding the movement of O for any change of length z in $C D$ we have, from (3) and (11),

$$d = z\frac{dy}{dl} = z\frac{P}{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In eq. (12), W is any arbitrary load, and P is the stress in the given member due to this load. The ratio P/W is constant. For convenience, therefore, W may be made unity. For such unit load, let $u =$ stress P, then we have, more briefly

$$d = z\,u, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and finally, for changes of lengths in several members,

$$D = \Sigma\, d = \Sigma\, z\,u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It is well to repeat that in (14) z is any given change of length in a member and u is the stress in the member due to a load of unity applied at the joint whose movement is desired and acting in the direction of this movement; it is the differential coefficient of the movement of the joint with respect to a change of length of the member.

Applying the second method of calculation to Fig. 2, the coefficient u is found by applying a load of unity at B. The coefficient u for $A B$ is the stress in $A B$ due to this unit load, $= 1 \times \frac{l_1}{y}$, and the coefficient for $C B = -1 \times \frac{l_2}{y}$, exactly as given in (6) and (7). In fact, if the methods of stress analysis for a single load W be traced back, using the principle of moments, it will be found that the two methods here given are practically identical, but the short cuts learned in stress analysis make the second method more expeditious in its application. The principle of "virtual velocities" is another form of the same thing.

To find the total movement in space of any joint, its movement in two directions (conveniently at right angles) must be found and the resultant movement determined.

210. Deflection Due to Stresses in the Structure from Applied Loads.—Suppose the changes in length z are due to stresses caused by a load of any kind. Let

S = stress in any member due to given load;
A = cross-section of member;
l = length of member;
E = modulus of elasticity;

Then $z = \frac{S\,l}{EA}$, and hence

$$D = \Sigma \frac{Sl}{EA} \cdot u. \quad \quad (15)$$

211. Deflection Due to Temperature Changes.—Suppose the changes of length z are due to temperature changes. Let

ω = coefficient of expansion; and
t = change of temperature in any member,

then

$z = \omega\, t\, l$, and hence

$$D = \Sigma\, \omega\, t\, l.\, u. \quad \quad (16)$$

212. Deflection Due to Looseness of Joints or Errors in Lengths.—If the effective length of any member when put in place is different from the calculated length, due to play in pin holes, errors in length or other cause, the resulting displacements may be obtained from eq. (14) by substituting for the various values of z the actual variations in length of the several members.

213. The Deflection Expressed as a Function of the Work of Distortion.—The work of distortion of a member under the stress S is equal to the distortion $S\,l/E\,a$, multiplied by one-half of the stress S; or, if K = work we have, for any member

$$K = \frac{S^2\, l}{2\, EA}. \quad \quad (17)$$

Differentiating this with respect to an arbitrary load W, applied as heretofore explained, we have

$$\frac{dK}{dW} = \frac{S\,l}{EA} \cdot \frac{dS}{dW},$$

and for all the members we may write

$$\frac{dK}{dW} = \Sigma \frac{Sl}{EA} \cdot \frac{dS}{dW}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This expression is the same as eq. (15), u being equal to the ratio of increment of stress to increment of load, $= dS/dW$. Hence the principle that

The deflection of a framed structure, due to any given system of loads, is equal to the differential coefficient of the internal work of distortion, taken with respect to a force applied at the point where the movement is desired.

214. Deflection of Structures Containing Members Subjected to Bending Moments.—The analysis of Arts. 207–209 relates only to members subjected to direct stress. If some or all of the members act as beams, the expression for deflection takes a different form although the general principle is the same.

Suppose one of the members of the structure of Fig. 3 acts as a beam, whose moments and deflections contribute to the movement of the point O. In this case we will proceed at once to get the internal work of distortion due to these moments and differentiate this with respect to an arbitrary load W, applied as before.

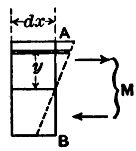


FIG. 4.

Let M = bending moment in the beam at any section due to the given loads, and consider an element dx of the beam at this section (Fig. 4). The stress f on a fibre of cross-section da, distant y from the neutral axis, $= \frac{My}{I} da$, and its elongation $= \frac{My}{I} \cdot \frac{dx}{E}$. The work of distortion is $\frac{1}{2} \frac{Myda}{I} \cdot \frac{Mydx}{EI} = \frac{1}{2} \frac{M^2 dx}{EI^2} y^2 da$. Integrating this across the section AB, we have, for the total work on the element dx,

$$\omega = \frac{M^2}{2EI^2} dx \int_A^B y^2 da = \frac{M^2}{2EI} dx. \quad . \quad . \quad . \quad (19)$$

Differentiating this with respect to load W applied at O we have

$$\frac{dK}{dW} = \frac{M dx}{EI} \cdot \frac{dM}{dW}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This is the deflection of the point O, due to the moment M in the element dx. The total effect of the bending moments throughout the

member in question is obtained by integrating (20) over the length of the member. Representing the total deflection due to the bending in this one member by d, as in eq. (3), we then have

$$d = \int_o^l \frac{M\,d\,x}{E\,I} \cdot \frac{d\,M}{d\,W} \quad \ldots\ldots \quad (21)$$

If $d\,W$ is made unity, then $d\,M$ is the moment at any point in the beam due to a load of unity placed at O. Call this moment m, corresponding to u of eq. (13). Then

$$d = \int_o^l \frac{M\,d\,x}{E\,I} \cdot m. \quad \ldots\ldots \quad (22)$$

For any number of members acting as beams

$$D = \Sigma\, d = \Sigma \int_o^l \frac{M\,d\,x}{E\,I} \cdot m. \quad \ldots\ldots \quad (23)$$

The complete expression for the deflection of a structure composed partly of members subjected to direct stress only and partly of members subjected to bending moments is therefore

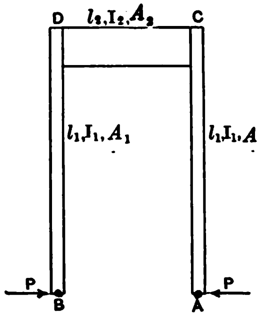


FIG. 5.

$$D = \Sigma \frac{S\,l}{EA} \cdot u + \Sigma \int_o^l \frac{M\,d\,x}{E\,I} \cdot m. \quad . \quad (24)$$

EXAMPLE.—A single example will be given of the application of eq. (24). Further applications to many special problems will be found in Part II.

Fig. 5 represents a portal frame with plate-girder top beam. Lengths, areas and moments of inertia are as shown. Find the lateral deflection of A with respect to B due to the forces P. The members $A\,C$ and $B\,D$ are beams, while $C\,D$ is both a beam and a strut.

For members $B\,D$ and $A\,C$, origin at B and C respectively,

$$M = P\,x \text{ and } m = x.$$

$$\int_o^l \frac{M\,d\,x}{E\,I} \cdot m = \frac{P}{E\,I_1} \int_o^{l_1} x^2\,d\,x = \frac{P\,l_1^3}{3\,E\,I_1}.$$

For $C\,D$

$$M = P\,l_1 \text{ and } m = l_1$$

$$\int_o^l \frac{M\,d\,x}{E\,I} \cdot m = \frac{P\,l_1^2\,l_2}{E\,I_2};$$

and for $C\,D$ as a strut

$$S = P,\ u = 1,\ \frac{S\,l}{EA} \cdot u = \frac{P\,l_2}{EA_2}.$$

Adding the several terms we have

$$D = 2 \cdot \frac{P l_1^3}{3 E I_1} + \frac{P l_1^2 l_2}{E I_2} + \frac{P l_2}{E A_2}.$$

Suppose $P = 4{,}000$ lbs., $l_1 = 30$ ft., $l_2 = 16$ ft., $I_1 = 2{,}000$ in.^{4}, $I_2 = 3{,}000$ in.^{4}, $A_2 = 15$ in.^{2}, and $E = 29{,}000{,}000$. We have $D = 2 \times .79 + .86 + .001 = 2.44$ in. The term .001 takes account of the effect of the direct compression in $C D$. It is very small compared to the effect of bending.

215. Calculation of the Deflection of a Pratt Truss.—Let it be required to determine the downward deflection of point d, Fig. 6, due

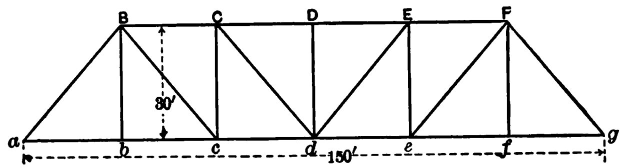


FIG. 6.

to a uniform load of 5,000 lbs. per ft., or 2,500 lbs. per ft. per truss. The calculations are given below in convenient tabular form. The lengths and gross cross-sections of the members are given in the table.

Member.	Length (l), Inches.	Stress (S), Pounds.	Area (A) Sq. In.	$\frac{S l}{E A}$	u	$\frac{S u l}{E A}$
$a B$	469	− 203,000	31.8	−.103	− .651	+.067
$B C$	300	− 208,500	31.8	−.068	− .833	+.056
$C D$	300	− 234,500	31.8	−.076	−1.250	+.095
$a b$	300	+ 130,000	26.4	+.051	+ .417	+.021
$b c$	300	+ 130,000	26.4	+.051	+ .417	+.021
$c d$	300	+ 208,500	39.5	+.055	+ .833	+.045
$B c$	469	+ 122,000	18.7	+.106	+ .651	+.069
$C d$	469	+ 40,600	10.0	+.066	+ .651	+.043
$B b$	360	+ 62,500	17.6	+.044	0	+. 0
$C c$	360	− 31,250	14.7	−.026	− .500	+.013
$D d$	360	0	11.8	0	0	0

$\Sigma = + .430$

The value of E is taken at 29,000,000 lbs. per sq. in. By reason of symmetry, only one-half of the truss is considered, the total deflection being twice the result so obtained. The value of u is the stress in a member due to one pound applied at d. Its sign in this case is the same as that of S for all members, so that the sign of $S\,u\,l/E A$ is always positive. The total deflection $= 2 \times 0.430 = 0.860$ in.

Suppose the horizontal movement of point *d* be required, the point *a* standing fast. Here the 1-pound load is to be applied at *d* and acting towards the right, reaction at *a*. The only members stressed will be *a b*, *b c* and *c d*, for each of which $u = 1$. We then have, briefly,

Member.	$\frac{Sl}{EA}$	u	$\frac{Sul}{EA}$
a b	+.051	+1.0	+.051
b c	+.051	+1.0	+.051
c d	+.055	+1.0	+.055

$\Sigma = +.157$

The horizontal displacement of *d*, due to the given load, is therefore 0.157 in. towards the right.

Again, suppose the temperature of the lower chord is 10° less than that of all the other members, what will be the resulting deflection of point *d* with reference to its position for uniform temperature conditions? The members to be considered are the lower chord members only, and these are to be assumed to be shortened by an amount equal to $\omega t l$, as in eq. (16). Take $\omega = .0000065$; $t = 10°$, $l = 300$. Then for each member, $\omega t l = .0000065 \times 10 \times 300 = .0195$ in. The value of u is as given in the large table. We have, then, for one-half the truss,

Member.	$\omega t l$	u	$\omega t l u$
a b	−.0195	+.417	−.00813
b c	−.0195	+.417	−.00813
c d	−.0195	+.833	−.01625

$\Sigma = -.03251$

The deflection of *d* is therefore upwards and in amount is equal to $2 \times .0325 = .065$ in.

216. Deflection Formula for a Pratt Truss.—For approximate values of the deflection of any style of truss we may obtain a formula which is readily evaluated, provided we may assume some average values for intensities of the tensile and compressive stresses. Thus, for a Pratt truss, single intersection, of an even number of panels (Fig. 7)

let p_t = average unit stress of tension members;
p_c = average unit stress of compression members;
E = modulus of elasticity for all members;
h = height of truss in inches;
d = panel length in inches;
n = number of panels in bridge.

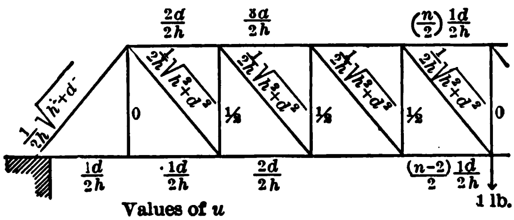


FIG. 7.

The values of u for each member are given in Fig. 7 in terms of the dimensions. We have then, from eq. (15),

$$\left.\begin{array}{lll}\text{For the Upper Chord,} & \Sigma \dfrac{Sl}{EA} \cdot u & = \dfrac{p_c d^2}{8Eh}(n+4)\ (n-2); \\ \text{For the Lower Chord,} & \text{``} & = \dfrac{p_t d^2}{8Eh}[n(n-2)+8]; \\ \text{For the Web Tension Members} & \text{``} & = \dfrac{p_t}{2Eh}(n-2)\ (h^2+d^2); \\ \text{For the End Posts} & \text{``} & = \dfrac{2p_c}{2Eh}(h^2+d^2); \\ \text{For the Verticals,} & \text{``} & = \dfrac{p_c h^2}{2Eh}(n-4). \end{array}\right\} \quad (25)$$

Whence for the whole truss the total deflection of the middle point for a full load is

$$D = \frac{p_t + p_c}{2Eh}\left[(n+2)\frac{nd^2}{4} + (n-2)\,h^2\right], \quad . \quad . \quad (26)$$

It will be noticed that in this case there is nothing to sum but u for each member, as grouped above in eqs. (25), p, l, and E being constant for all the members of a group. Also for such members as give a value of $u = 0$, as for the middle vertical and the end hanger, they are of

course omitted, or rather count for nothing in the summation. This means that these two members do not in any way contribute to the deflection of the middle point.

Applying this formula to the example of Art. 215, we may assume p_t and $p_c = 6{,}000$ lbs. per sq. in., approximately. Substituting in (26) we derive the value $D = 0.77$ in.

Since the maximum stresses in the web members do not occur for full load the value of the unit stress to be assumed for these members for a fully loaded structure will be less than their working stresses.

217. *Relative Deflection from Web and Chord Stresses.*—By adding the deflection increments due to web members and those due to chord members, we may obtain,

$$\left.\begin{array}{ll}\text{For Chord,} & \Sigma \dfrac{Sl}{EA}\cdot u = \dfrac{d^2}{8Eh}\left[(n(n-2)+8)\,p_t + (n+4)(n-2)\,p_c\right]; \\[2ex] \text{For Web,} & \quad `` \quad = \dfrac{1}{2Eh}\left[(n-2)(h^2+d^2)p_t + ((n-2)h^2+2d^2)p_c\right].\end{array}\right\} \quad (27)$$

If we should assume that the average stress in the compression members is 0.7 that in the tension members or $p_c = 0.7\,p_t$, we may write,

$$\left.\begin{array}{ll}\text{For Chords,} & \Sigma \dfrac{Sl}{EA}\cdot u = \dfrac{p_t d^2}{8Eh}(1.7\,n^2 - 0.6\,n + 2.4); \\[2ex] \text{For Web,} & \quad `` \quad = \dfrac{p_t}{2Eh}\left[(1.7\,n - 3.4)\,h^2 + (n - 0.6)\,d^2\right].\end{array}\right\} \quad . \quad (28)$$

Whence

$$\frac{\text{Deflection from web}}{\text{Deflection from chords}} = \frac{(6.8\,n - 13.6)\left(\dfrac{h}{d}\right)^2 + 4\,n - 2.4}{1.7\,n^2 - 0.6\,n + 2.4}. \quad . \quad . \quad (29)$$

This ratio increases as $\dfrac{h}{d}$ increases, and decreases as n, the number of panels, or length of bridge, increases.

For a Pratt truss bridge of 200 feet span, of ten panels, and a height of 30 feet, this fraction becomes 80/83 or 96.4 per cent. For a truss of eight panels and same span and height the ratio is 115 per cent.

That is to say, for such span lengths and for the assumptions made, the deflection from web distortion is about equal to that from chord distortion.

This is quite contrary to the condition in a solid beam where the shearing (or web) distortions are small as compared to the moment (or flange) distortions. This is due primarily to the fact that in the truss there is no surplus material in the web as is generally the case in the beam.

A common method of calculating the deflection of trusses, and one formerly much used, is to consider them as beams and apply the formula for the deflection of a beam, using a moment of inertia of the truss as determined from the chord members. Obviously, in view of the foregoing examples, the results thus obtained are quite inaccurate and too small, except in the case of trusses in which the number of panels is large and the height relatively small.

218. Height of Truss for Maximum Stiffness and Economy.—We may differentiate equation (26) for h variable and find

$$\frac{dD}{dh} = \frac{p_c + p_t}{2E}\left[-\frac{(n+2)\,n\,d^2}{4\,h^2} + (n-2)\right],$$

$$= \frac{p_c + p_t}{8\,E\,h^2}\left[(n-2)\,4\,h^2 - (n+2)\,n\,d^2\right]. \quad . \quad . \quad (30)$$

Placing this quantity equal to zero, and solving for h, we find the height of truss which will give a minimum deflection to be

$$h^2 = \frac{n+2}{n-2}\cdot\frac{n\,d^2}{4},$$

or

$$h = \frac{d}{2}\sqrt{\frac{n+2}{n-2}\cdot n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

From eq. (31) we find that for a minimum deflection, or for a maximum stiffness, for given working-unit stresses, the height of this stiffest truss has the following values:

TABLE OF HEIGHT OF PRATT TRUSSES OF MAXIMUM STIFFNESS.

Height	1.73d	1.73d	1.82d	1.94d	2.05d	2.16d	2.27d	2.37d	2.42d
No. of panels	4	6	8	10	12	14	16	18	20
$\frac{\text{Length}}{\text{Height}}$	2.3	3.4	4.3	5.3	5.9	6.7	7.1	7.7	8.3

It can also be shown in a general way that the stiffest truss contains the least material for given working stresses. For, considering the truss fully loaded and all members stressed to a value p per square inch (this will be only approximately true), the total work of distortion in the members will be $\Sigma \frac{1}{2} pA \times \frac{pl}{E} = \frac{1}{2} \Sigma \frac{p^2Al}{E}$. The external work will be $\frac{1}{2} \Sigma W D, = \frac{1}{2} W \Sigma D$, where W = joint load and D = deflection of a loaded joint. Therefore, $\Sigma D = \dfrac{\Sigma \frac{p^2Al}{E}}{W}$. For greatest stiffness ΣD is a minimum, but this requires a minimum value of $\Sigma \frac{p^2Al}{E}$, or, since p and E are constants, a minimum value of $\Sigma A l$, which is the volume of the truss. Hence the truss of minimum volume is also the stiffest truss.

It would appear, therefore, that maximum economy would require proportions about as given in the table. In practice, the heights adopted are somewhat less than those given, especially for the smaller number of panels, and this is as it should be. In the analysis, a fixed value of working stress was assumed. In fact, however, the working stresses for compression members decreases with increased length or decreased cross-section. Inasmuch as an increase in height increases the ratio of length to cross-section for all compression members and so reduces their working stresses, true economy of metal calls for a less height than the calculated values.

219. Graphical Method of Determining Displacement of the Joints of a Framed Structure. (Williot Diagrams.)—When the deformations of the individual members have been computed the relative movement of all the joints of the structure may be determined by a very simple graphical method. Consider the triangle $a\,b\,c$, Fig. 8. Suppose the members $a\,b$ and $b\,c$ are elongated and $a\,c$ compressed a certain known amount. Required the new positions of a, b and c, it being assumed that the centre point of $a\,b$ stands fast and that $a\,b$ does not change its direction.

Lay off one-half the elongation of $a\,b$ from b and one-half from a; a' and b' are the new positions of a and b. Then with a' and b' as

centres and radii equal to the new lengths of $a\,c$ and $b\,c$ the new position c' of c, is determined. These new lengths are shown as b' 2 and a' 4, the changes of lengths being 1–2 and 3–4. In the same manner the new position of other joints in the frame might be determined. Practically, however, the displacement cannot be determined in this way, as the changes in length are too small, compared to the lengths of the members, to be represented in this manner. The same end will be attained if the changes in lengths only are plotted, omitting the members themselves. Thus in Fig. 8, if from point c the length c–1 is made equal to $b\,b'$ and c–3 equal to $a\,a'$, and the line 1–2 drawn parallel to $b\,c$ and equal to its elongation, and 3–4 drawn parallel to $a\,c$ and equal to its compression; then if from 2 and 4 perpendiculars be erected

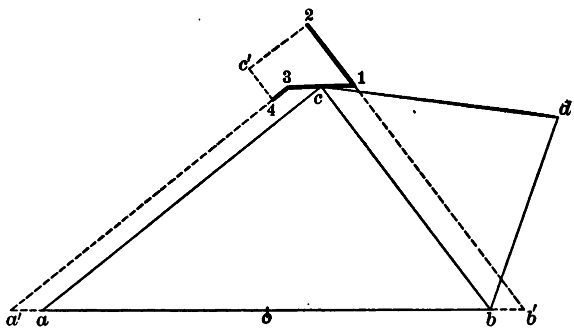


FIG. 8.

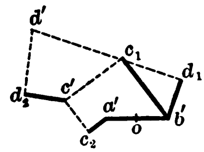


FIG. 9.

(the radius of the circle being relatively very long), the intersection c' will represent the new position of c, and so on.

For convenience, a separate diagram, Fig. 9, should be constructed. Lay off first $a'\,b'$ equal to the elongation of $a\,b$, and on a line parallel to $a\,b$; then $b'\,c_1$, equal to the elongation of $b\,c$, and $a'c_2$ equal to the shortening in $a\,c$. Perpendiculars at points c_1 and c_2 determine c'. Then for joint d lay off $c'\,d_2$ and $b'\,d_1$ equal to the given deformations in members $c\,d$ and $b\,d$; and, again, perpendiculars at points d_1 and d_2 will determine d', which gives the movement of joint d relative to all other joints; and so on. That is, the points a', b', c', d', etc., thus determined in Fig. 9, referred to point o, give the amounts and directions of the actual movement in space of the joints of the structure, the point o and the line $a\,b$ standing fast. If the point o and the line

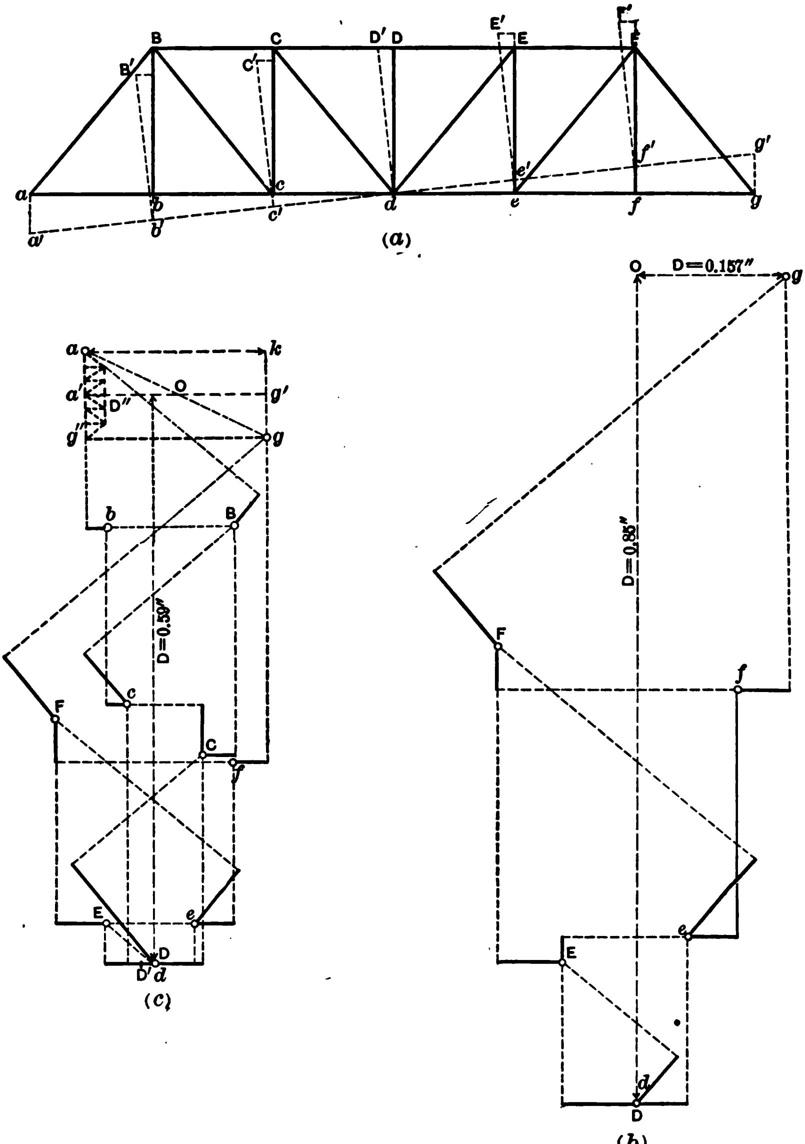


FIG. 10.

$a\ b$ shifts in position then the diagram still gives the correct relative movements in amount but not in direction.

A great advantage of the graphical method over the algebraic method of Art. 209 lies in the fact that the former gives, in a single diagram, the displacements of all joints, while the latter applies to but one joint at a time, and in but one direction. Where the displacement of a single point only is needed the analytical method is very satisfactory.

Example 1.—A displacement diagram will be drawn for the truss of Art. 215. The centre vertical will be a convenient place to commence the diagram. Symmetrical loads require the construction of one-half the diagram only; Fig. 10 (*b*) gives the complete construction. The vertical distance $O\ d$, from point $D\ d$ to point g, gives the vertical deflection, and the distance $O\ g$ is one-half of the total elongation of $a\ g$.

Example 2.—Same example with unsymmetrical load. Assume joints d, e and f loaded with 62,500 lbs. each. Assuming as before that $D\ d$ stands fast, the diagram, Fig. (*c*), is constructed. The resulting position of points a and g gives the correct relative movements of these points referred to $D\ d$, but not the absolute movements, as $D\ d$ does not remain vertical.

To get the true direction of the displacements we make use of the fact that points a and g should have no vertical motion. The vertical component, $g\ k$, Fig. (*c*), of the displacement of g, relative to a, indicates therefore the amount by which the truss has been revolved about d in assuming $D\ d$ to remain vertical. This component measures 0.087 in. To get the true displacements, considering the line joining the abutments a and g to stand fast, we may rotate the entire truss, Fig. (*a*), through a small angle determined by making $g\ g' = \frac{1}{2} \times .087 = .0435$ in. This may be done in Fig. (*a*) to an exaggerated scale as shown by the dotted lines. The actual angle is very small and therefore perpendiculars are used instead of arcs of circles to give the direction of motion. The point D will move to the left a distance $D\ D' = g\ g' \times \dfrac{30}{75} = 0.0174$ in., and other points will move as shown in the figure. All lower chord points move, vertically, distances proportional to their distances from d; all upper chord points move the same amounts vertically and the distance 0.0174 in. horizontally. These

movements may be applied to Fig. (c) as corrections to the positions there given; or the point D' may be marked in Fig. (c) and a new diagram drawn with $d\,D'$ as a correct base. The same results will be obtained by either method.

Instead of correcting the position of each point as above described in order to get the true movements, the same result may be more readily obtained by making all the corrections at the fixed point of reference and measuring all distances from the several new points thus obtained. Thus, if a is the fixed point, then to get the movement of g, for example, instead of laying off the corrections $g\,g'$ and $a\,a'$, we lay off the whole distance $a\,g''$ downward from a. Then $g''\,g$ is the desired movement. Likewise for point D lay off $a'\,D''$ to the right and equal to $D\,D'$ of Fig. (a), then the distance from D'' to D of Fig. (c) is the desired movement. If the several points so laid off be joined there will result a figure exactly similar to the given truss, with lower chord equal to $a'\,g''$.

220. Maxwell's Law of Reciprocal Deflections; Influence Lines for Deflection.—If the vertical components of the movements of the lower chord joints, as corrected in Fig. 10 (c), be laid off from a horizontal axis the resulting curve is the *deflection curve* for the

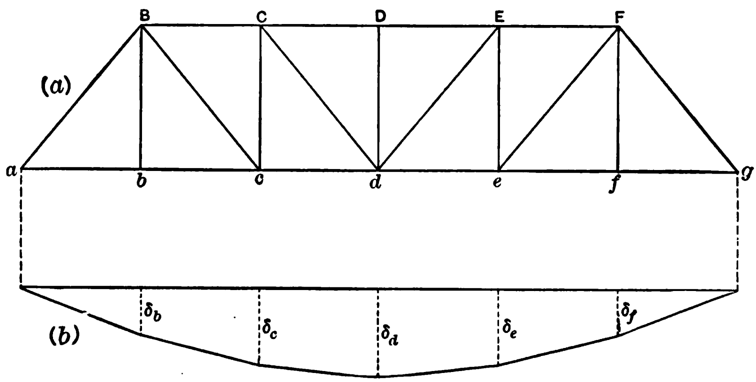


FIG. 11.

lower chord joints. A similar curve may be drawn for the upper chord joints.

Suppose now that a displacement diagram similar to Fig. 10 (b) be constructed for a load of one pound placed at d. Let Fig. 11 (b)

be the deflection curve for the lower chord joints, constructed from this diagram. Then any ordinate, as δ_c, is the deflection of joint c due to a load of one pound placed at d. Expressed algebraically the deflection is given by the equation

$$\delta_c = \Sigma \frac{S u l}{EA}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

in which S = stress in any member due to the given loads (one pound at d) and u = stress in any member due to one pound placed at c. Noting that in this case S is the stress due to a one-pound load at d, it will be seen that this expression for deflection at c is identical with the expression for the deflection at d for one pound placed at c, the quantities S and u being interchanged. Hence the important principle that

The deflection at d for a one-pound load acting at any other point c is equal to the deflection at c for a one-pound load acting at d.

For a load P placed at c the deflection at d will be $P\,\delta_c$, and for any number of joints loaded the deflection at d is given by the general expression

$$D_d = \Sigma P \delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The deflection curve of Fig. 11 (*b*) is therefore the *influence line* for deflection at d, from which the deflection at this point can readily be calculated for any given loading. It is to be seen that such an influence line requires the construction of but a single Williot diagram, while the analytical method would need to be applied separately for each individual joint.

Deflection influence lines are of special value in determining stresses in redundant members and reactions of swing bridges, arches and other structures having redundant supports. Their use for such purposes is further explained in Art. 223 and in Part II.

Section II.—Stresses in Redundant Members

221. General Principles.—Redundant members of a framework are those members which are in excess of the minimum number required to make the framework a rigid structure, that is, one which is fixed in form except for small changes due to stress. The triangle is the truss

element, and the least number of members required to fix the relative position of a given number of points may be determined by conceiving the truss made up of triangles. The first elemental triangle has three sides and fixes three points. To fix each additional point requires two additional sides, or members; hence if m = total number of points and n = total number of necessary members, we have the relation

$$n - 3 = 2\,(m - 3), \quad \text{whence } n = 2\,m - 3. \quad . \quad . \quad (1)$$

Whether a truss has redundant members can usually be ascertained by inspection, but in complicated cases it may be convenient to employ the foregoing rule, noting carefully, however, that every part of the structure has sufficient members for the stability of that part.

Applying this rule to the truss of Fig. 15, we have m = 14 and hence n = 25. The actual number of members is 26, hence there is one redundant member. This is a familiar case. Another common example of redundancy is where two diagonals are used in the same quadrilateral. Usually they are both rods or bars and incapable of carrying compression, and the assumption that they act only in tension enables their stress to be determined by the methods of statics. If they are built to take both tension and compression, the stresses cannot thus be determined.

Every redundant member introduces one unknown quantity in excess of the number determinable by statics; hence a structure is said to be *singly* or *doubly* indeterminate as it has one or two redundant members, etc.

The calculation of the stresses in structures with redundant members is made possible by determining the relation between the distortion of the necessary members and of the redundant members. For this purpose the general formula for deflection, eq. (15), may be utilized. It is

$$D = \Sigma \frac{S\,u\,l}{A\,E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which D = deflection or movement of any joint in any given direction due to any loading, S = total stress in any member due to this loading, u = factor of reduction = numerically the stress in the member due to one pound applied at the point whose movement is desired and acting in the given direction, and l, A, and E are length, cross-section, and modulus of elasticity of the member.

222. Stresses in Structures Having a Single Redundant Member.—Let $A\,B$, Fig. 12, represent any structure loaded in any manner (at joints only), and having one redundant member. In this case any member may be taken for the redundant member; we will take member 6. The truss is hinged at A and supported on rollers at B. Let S_1, S_2, S_3, etc., be the stresses in the several members, as yet unknown. S_6 is the stress in the redundant member. Represent lengths, cross-sections, and moduli of elasticity by l_1, l_2, etc., A_1, A_2, etc., and E_1, E_2, etc., respectively.

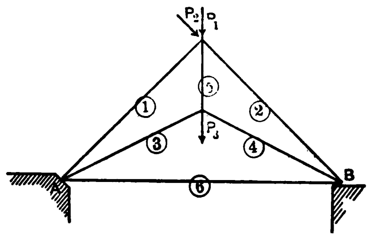


FIG. 12.

The desired elastic relation can be obtained as follows:

Cut the member 6 very close to one end, as at A, Fig. 13. Let A' represent the free end of the member. Consider a force S_6, equal to the stress in 6, applied to joint A and also to the end of the member at A'. The stresses in the various members will not be disturbed by this operation. If now the deflection of the point A', with reference to A be calculated by means of the distortions of the various members, 1–6, it must equal zero, the point A' being very near A. This deflection is given by the formula $D = \Sigma_1^6 \frac{S\,u\,l}{EA}$, in which Σ^6_1 represents a summation for members 1–6. Hence we have

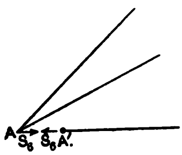


FIG. 13.

$$\Sigma_1^6 \frac{S\,u\,l}{EA} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the desired elastic relation. By expressing S in terms of the external forces and the stress in the redundant member the latter may be calculated.

To put this into convenient form for calculations, the stress S in any member may be considered as made up of two parts: (*a*) a stress S' which would be caused by the external loads with the redundant member 6 removed, and (*b*) a stress S'' due to forces applied at A and A' equal to S_6. The first part, S', is readily calculated by the usual methods of statics. The second part, S'', can also be calculated

when S_6 is known. Noting that u is the stress in any member due to 1 pound acting towards the left at A' (reaction at A is also 1 pound), it is evident that S'' is equal to $S_6 u$. Hence we may write the general relation

$$S = S' + S_6 u. \qquad (4)$$

For the redundant member, No. 6, $S' = 0$ and $u = 1$.

Substituting in eq. (3), we have

$$\Sigma_1^6 \frac{S u l}{EA} = \Sigma_1^5 \frac{S' u l}{EA} + S_6 \Sigma_1^6 \frac{u^2 l}{EA} = 0.$$

Whence

$$S_6 = -\frac{\Sigma_1^5 \dfrac{S' u l}{EA}}{\Sigma_1^6 \dfrac{u^2 l}{EA}}. \qquad (5)$$

In this expression all quantities are readily calculated. Careful attention must be paid to sign, tension being considered plus and compression minus. The stress u is the stress in any member for 1 lb. *tension* in the redundant member.

The value of S_6 being obtained from (5), the stress in any member is readily got from (4), the value S' and u being already calculated. If E is constant it may be omitted from the calculations.

The form of expression is evidently the same no matter how many necessary members there may be. If n is the number of such members and r is the redundant member, we may write the more general expression

$$S_r = -\frac{\Sigma_1^n \dfrac{S' u l}{EA}}{\Sigma_1^{n+1} \dfrac{u^2 l}{EA}}, \qquad (6)$$

in which it is to be noted that S' is the stress in any of the n members due to the given loads, with the redundant member removed, and u is the stress due to a tension of 1 pound in the redundant member. In the denominator the redundant member itself must be included, the value of u being unity.

EXAMPLE 1.—Let it be required to calculate the stresses in the structure of Fig. 12, assuming the following data: Span length = 30 ft.; height = 15 ft.; length of member 5 = 7.5 ft.; P_1 = 40,000 lbs.; P_2 = 10,000 lbs.; P_3 = 30,000 lbs. The cross-sections

will be assumed at 4 sq. in. for 1 and 2, 3 sq. in. for 3, 4, and 6, and 2 sq. in. for 5. The calculations are given in a convenient form in the table below. The first three columns contain the given data. The stresses S' are then calculated with member $A\ B$ removed;

Member.	l	A	S'	u	$\frac{u\,l}{A}$	$\frac{S'\,u\,l}{A}$	$\frac{u^2\,l}{A}$	$S_6\,u$	S
1	255	4	− 109.2	+ 1.41	+ 89.5	− 9,760	+ 126	+ 68.0	− 41.2
2	255	4	− 119.2	+ 1.41	+ 89.5	− 10,650	+ 126	+ 68.0	− 51.2
3	201	3	+ 94.0	− 2.23	− 149.5	− 14,050	+ 334	− 107.5	− 13.5
4	201	3	+ 94.0	− 2.23	− 149.5	− 14,050	+ 334	− 107.5	− 13.5
5	90	2	+ 114.2	− 2.00	− 90.0	− 10,280	+ 180	− 96.4	+ 17.8
6	360	3		+ 1.00	+ 120.0		+ 120		+ 48.2
						− 58,790	+ 1,220		

$$S_6 = -\frac{-58{,}700}{1{,}220} = +48.19.$$

also the stresses u due to 1 pound tension in A B. It is then convenient to calculate $\frac{u\,l}{A}$, and from this the values of the next two columns. Then from the summations of these values S_6 is obtained. Then the values of $S_6 u$, and finally the values of S from eq. (4).

EXAMPLE 2.—Fig. 14 shows the common case of two diagonals of a panel of a lateral system. Members 1 and 2 are chord members, 3 and 4 are struts, and 5 and 6 are diagonals, assumed for the present to be able to take either tension or compression. A shear of 8,000 lbs. is assumed to exist in the panel, and other forces are introduced to give equilibrium. The other data are given in the table below. Member 6 is assumed to be the redundant member. The cross-sections of 1 and 2 are so large that they may be assumed as rigid compared to 5 and 6. The result shows that 6 receives a compression of 6.7 and 5 a tension of 6.2. If members 3 and 4 were also assumed rigid, the resulting stresses in 5 and 6 would be numerically equal, each member carrying one-half of the shear; and for all practical purposes this may always be assumed to be the case. A common example of stiff diagonal bracing is the usual cross-bracing between plate girders.

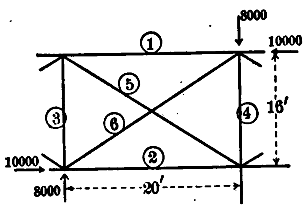


FIG. 14.

Member.	l	A	S'	u	$\frac{u\,l}{A}$	$\frac{S'\,u\,l}{A}$	$\frac{u^2\,l}{A}$	$S_6\,u$	S
1	240	large	− 10.0	− .783	0	0	0	+ 5.3	− 4.7
2	240	large	− 10.0	− .783	0	0	0	+ 5.3	− 4.
3	192	6	− 8.0	− .625	− 20.0	+ 160	+ 12.5	+ 4.3	− 3.7
4	192	6	− 8.0	− .625	− 20.0	+ 160	+ 12.5	+ 4.3	− 3.7
5	307	1	+ 12.9	+ 1.0	+ 307.0	+ 3,960	+ 307.0	− 6.7	+ 6.2
6	307	1		+ 1.0	+ 307.0		+ 307.0		+ 6.70
						+ 4,280	+ 639.0		

$$S_6 = -\frac{+4{,}280}{639.0} = -6.70.$$

If 5 and 6 are tie-rods and are given an initial tension, then the resulting stress in each tie-rod will equal the initial tension plus the stresses above given. As soon as the com-

pressive stress in 6 equals the initial tension, then this rod becomes idle and the other carries the entire shear, as usually assumed. A similar condition exists with reference to the counters of a truss, excepting that the areas of the two diagonals are not usually equal, and the shear is carried approximately in proportion to their areas.

EXAMPLE 3.—A complete analysis will be made of the double triangular truss of Fig. 15. To do this it is only necessary to calculate the stresses in all the members for a load of unity placed successively on each of the three joints, d, e, and f. From symmetry, the stresses for loads on b and c will then also be known. Knowing then the stresses due to a load of unity at each joint, the effect of any combination of loads can be determined as in any other case. Influence lines can also be drawn and used for getting maximum stresses. The three sets of computations are all given in the table following. The member $A\ a$ is taken as the redundant member in each case, and for this member $\frac{l}{A} = 40$. The several values of S' due to the several joint loads are denoted by S_f', S_e' and S_d', and the final stresses by S_f, S_e and S_d.

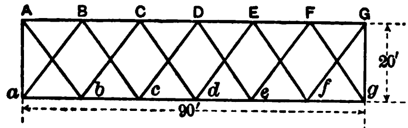


FIG. 15.

From the several summations we have

$$S_r \text{ (load at } f) = -\frac{+254.5}{1,516} = -.167;\quad S_r \text{ (load at } e) = -\frac{+49.3}{1,516} = -.032;$$

$$S_r \text{ (load at } d) = -\frac{+757.9}{1,516} = -.500.$$

The stresses in the other members are then found as before. They are given in the last three columns of the table.

The effect of loading any part of the bridge can now be ascertained, and comparisons made with the approximate method of calculation given in Chapter IV, based on the assumption of independent web systems. We first note that a load at d causes no stress in the web members $a\ B$, $B\ c$, etc., and these results are therefore exactly the same as obtained by the approximate method. Furthermore, when the truss is loaded symmetrically in any manner it will be found that the stresses obtained by the two methods are exactly the same. Hence for maximum chord stresses and maximum stresses in $a\ B$ and $A\ b$ the usual method gives correct results. For unsymmetrical loading, however, there is a small error in the assumption of independent systems. Thus it is seen that a load of unity at e causes stresses of .041 in each web member of the web system $a\ B\ c$, etc. For a unit load at f, the effect is too small to be noticed with the precision of calculation here employed. The true maximum stress in $C\ d$, for example, is found by adding the stresses for loads at d, e, and f. This sum is $+ .208 + .041 + .625 = + .874$ for unity loads. By the usual method of calculation it is $+ .833$, giving an error of .041, about 5 per cent. For member $B\ c$, joints c, d, e, and f are loaded. Stress for load at c = stress in $e\ F$ for load at $e = + .874$. Total in $B\ c$ therefore $= + .874 + 0.0 + .376 = + 1.25$. The other method also gives 1.25. It is evident from these calculations that the approximate method gives results sufficiently accurate.

223. Graphical Method of Calculation.—*Influence Lines for Stresses in Redundant Members.*—The graphical method of calculating deflec-

Member.	l	A	u	$\frac{ul}{A}$	S'_f	S'_e	S'_d	$\frac{S'_f ul}{A}$	$\frac{S'_e ul}{A}$	$\frac{S'_d ul}{A}$	$\frac{u^2 l}{A}$	S_f	S_e	S_d
$a\,b$	180	6	+ .75	+ 22.5	+ .125	+ .25	+ .375	+ 2.81	+ 5.62	+ 8.44	16.9	+ .007	+ .226	0
$b\,c$	"	6	− .75	− 22.5	+ .125	+ .25	+ .375	− 2.81	− 5.62	− 8.44	16.9	+ .250	+ .274	+ .75
$c\,d$	"	8	+ .75	+ 16.9	+ .375	+ .75	+ 1.125	+ 6.34	+ 12.68	+ 19.01	12.7	+ .250	+ .726	+ .75
$d\,e$	"	8	− .75	− 16.9	+ .375	+ .75	+ .375	− 6.34	− 12.68	− 6.34	12.7	+ .500	+ .774	+ .75
$e\,f$	"	6	+ .75	+ 22.5	+ .625	+ .50	+ 1.125	+ 14.06	+ 11.25	+ 25.32	16.9	+ .500	+ .476	+ .75
$f\,g$	"	6	− .75	− 22.5	− .125	+ .50	− .375	+ 2.81	− 11.25	+ 8.44	16.9	+ .750	+ .524	.0
$A\,B$	"	6	+ .75	+ 22.5	0	0	0	0	0	0	16.9	− .125	− .024	− .375
$B\,C$	"	6	− .75	− 22.5	− .25	− .50	− .75	+ 5.62	+ 11.25	+ 16.88	16.9	− .125	− .476	− .375
$C\,D$	"	8	+ .75	+ 16.9	− .25	− .50	− .75	− 4.22	− 8.45	− 12.68	12.7	− .375	− .524	− 1.125
$D\,E$	"	8	− .75	− 16.9	− .50	− 1.0	− 1.50	+ 8.45	+ 16.90	+ 25.35	12.7	− .375	− .976	− 1.125
$E\,F$	"	6	+ .75	+ 22.5	− .50	− 1.0	0	− 11.25	− 22.50	0	16.9	− .625	− 1.024	− .375
$F\,G$	"	6	− .75	− 22.5	− .75	0	− .75	+ 16.88	0	+ 16.88	16.9	− .625	+ .024	− .375
$a\,B$	300	6	− 1.25	− 62.5	− .208	− .417	− .625	+ 13.00	+ 26.05	+ 39.07	78.1	0	− .376	0
$B\,c$	"	6	+ 1.25	+ 62.5	+ .208	+ .417	+ .625	+ 13.00	+ 26.05	+ 39.07	78.1	0	+ .376	0
$c\,D$	"	3	− 1.25	− 125.0	− .208	− .417	− .625	+ 26.00	+ 52.10	+ 78.14	156.2	0	− .376	0
$D\,e$	"	3	+ 1.25	+ 125.0	+ .208	+ .417	+ .625	+ 26.00	+ 52.10	+ 78.14	156.2	0	+ .376	0
$e\,F$	"	6	− 1.25	− 62.5	− .208	+ .833	− .625	+ 13.00	− 52.10	+ 39.07	78.1	0	+ .874	0
$F\,g$	"	6	+ 1.25	+ 62.5	+ .208	− .833	+ .625	+ 13.00	− 52.10	+ 39.07	78.1	0	− .874	0
$A\,b$	"	6	− 1.25	− 62.5	0	0	0	0	0	0	78.1	+ .208	+ .041	+ .625
$b\,C$	"	6	+ 1.25	+ 62.5	0	0	0	0	0		78.1	− .208	− .041	− .625
$C\,c$	"	3	− 1.25	− 125.0	0	0	0	0	0	0	156.2	+ .208	+ .041	+ .625
$d\,E$	"	3	+ 1.25	+ 125.0	0	0	+ 1.25	0	0	+ 156.25	156.2	− .208	− .041	+ .625
$E\,f$	"	6	− 1.25	− 62.5	0	0	− 1.25	0	0	+ 78.12	78.1	+ .208	+ .041	− .625
$f\,G$	"	6	+ 1.25	+ 62.5	+ 1.25	0	+ 1.25	+ 78.13	0	+ 78.12	78.1	+ 1.042	− .041	+ .625
$G\,g$	240	6	− 1.00	− 40.0	− 1.00	0	− 1.00	+ 40.00	0	+ 40.00	40.0	− .833	+ .032	− .50
$A\,a$	"	6	+ 1.00	+ 40.0							40.0			
								+ 254.48	+ 49.30	+ 757.91	+ 1515.6			

tions, explained in Art. 220, can readily be applied to the solution of stresses in redundant members. The method of calculation will be explained by the solution of Example 2, of the preceding Article.

To apply this method the displacement diagram will be drawn for the truss for a load of one pound applied at the lower end of the redundant member $A\ a'$, the member being cut just above a. No other loads are to be considered acting. The stresses in the several members are given by the values of u in the preceding table and the deformations (multiplied by E), are given by the values $\frac{u\,l}{A}$. The term E being constant may be neglected and the values of $\frac{u\,l}{A}$ used directly. It will be convenient to begin the diagram at g. The complete diagram is given in Fig. 16 (b), The correction diagram is shown in dotted lines, assuming point a to be the fixed point. The vertical deflection of a', Fig. (a), is measured by the distance $a\ a'$ of Fig. (b). It is found to be 1,516 units, checking with the value of $\Sigma\,\frac{u^2l}{A}$ of the table. The measured vertical deflections of the lower chord joints are as follows:

Joint.	Deflection.
b	1,262 down
c	50 up
d	756 down
e	50 down
f	254 down

Plotting these in Fig. (c) gives the deflection influence line $a''\ b''\ g'$ for the deflection of point a' of Fig. (a).

To utilize this diagram for determining stresses in $A\ a'$ it is first to be noted that as the ordinate δ_a gives the *downward* deflection of point a' for a one-pound load applied at a', the length of this ordinate represents also the *upward* deflection of a' for a one-pound load acting *upward* at a'. Consider now a load of one pound acting at b with member $A\ a$ cut as before. The downward deflection of a' caused by this load is given by the ordinate δ_b. If a sufficient *upward* force now be applied at a' to reduce the deflection at this point to zero, the amount of such upward force will be equal to δ_b/δ_a. This upward

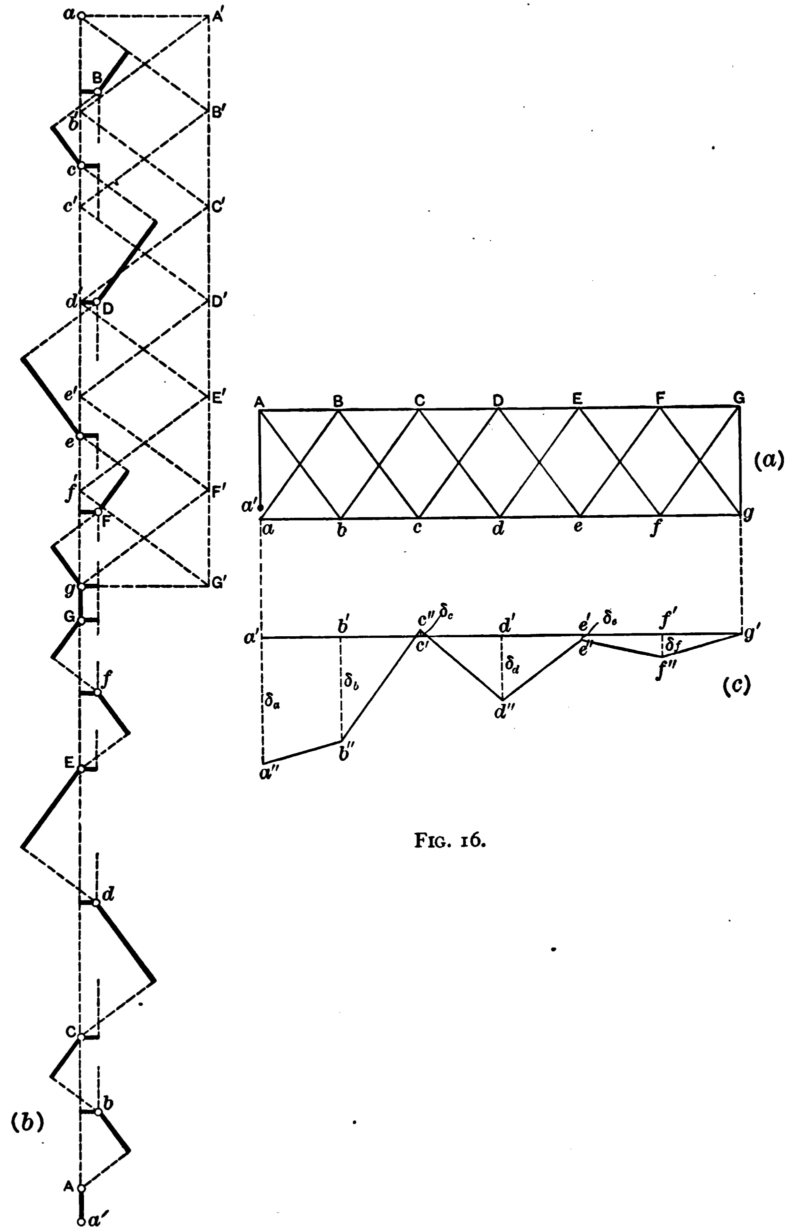

A
B
C
D
E
F
G
a′
a
b
c
d
e
f
g
(a)
(b)
(c)
a′
b′
c″
c′
d′
e′
e″
f′
f″
g′
d″
b″
a″
δa
δb
δc
δd
δe
δf
A′
B′
C′
D′
E′
F′
G′

FIG. 16.

force is evidently the *compression* which would be caused in the redundant member $A\ a$ for a one-pound load at b. In a similar manner it can be shown that the stress in the redundant member due to a one-pound load at c is a tension equal to δ_c/δ_a (the ordinate δ_c showing an *upward* deflection of a'). Hence in general the stress in the redundant member caused by a one-pound load at any point is equal to δ/δ_a, where δ = deflection of given point for a one-pound load acting at the cut end of the redundant member. Due attention must be paid to sign, it being noted that when the deflections δ and δ_a are of like sign, the stress in the redundant member is compression, assuming the unit loads applied as here explained.

Finally for any number of joint loads P, the total stress in the redundant member is given by the expression:

$$S_r = -\frac{\Sigma P \delta}{\delta_a}. \qquad (7)$$

This equation is, in fact, identical with eq. (5) of Art. 222.

In the present example, dividing the joint deflections by 1,516, we have the following stresses for one-pound loads:

	Stress in $A\ a$
Load at b	− 0.832
" " c	+ 0.033
" " d	− 0.500
" " e	− 0.033
" " f	− 0.167

These results are seen to agree very closely with those obtained by the analytical method.

It is to be noted that if the diagram of Fig. 16 (c) be replotted to a scale such that δ_a = unity the curve becomes an *influence line for stress in the redundant member.*

224. *Stresses Due to Errors in Length.*—In a structure with redundant members, if such members are not of exactly the correct length, initial stresses will be caused throughout the structure. The effect of a given error in length may be calculated. In Fig. 8, if member 6 is made too short by an amount x, then the deflection given by eq. (3) must be equal to x. Or, in general,

$$\Sigma \frac{S\,u\,l}{EA} = x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As before, $S = S' + S_r\,u$, but in this case S' is zero. Substituting in (8), we derive

$$S_r = -\frac{x}{\Sigma^{n+1}{}_1 \dfrac{u^2\,l}{EA}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

225. Stresses in Structures having Two or More Redundant Members.—The foregoing analysis has dealt with but a single redundant member. Suppose there are two, the number of necessary members being n. Let S_r and S_{r+1} be the stresses in the redundant members, l_r and l_{r+1} their lengths, etc. S_1, S_2, etc., are the actual stresses in the other members. Consider, first, the movement at the end of member r. As before, this movement may be expressed in terms of the distortions of the necessary members and of member r. The equation is (from (3))

$$\Sigma^n{}_1 \frac{S\,u\,l}{EA} + \frac{S_r\,l_r}{E_r A_r} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which u is the stress in any of the n members due to a one-pound tension in member r, *the other redundant member being removed.* Likewise we can write a similar expression for the distortion in the direction of member $r + 1$, and have

$$\Sigma^n{}_1 \frac{S\,v\,l}{EA} + \frac{S_{r+1}\,l_{r+1}}{E_{r+1}A_{r+1}} = 0, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which v is the stress in any of the n members due to a 1-pound tension in member $r + 1$, *member r being removed.*

In this case we may also write

$$S = S' + S_r\,u + S_{r+1}\,v, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where S' is the stress due to the external loads, both redundant members being removed. From these equations we may derive the two similar equations.

$$S_r\left(\frac{l_r}{E_r A_r} + \Sigma_1{}^n \frac{u^2\,l}{EA}\right) + S_{r+1}\,\Sigma \frac{u\,v\,l}{EA} = -\Sigma_1{}^n \frac{S'\,u\,l}{EA}; \quad . \quad (13)$$

$$S_{r+1}\left(\frac{l_{r+1}}{E_{r+1}A_{r+1}} + \Sigma_1{}^n \frac{v^2\,l}{EA}\right) + S_r\,\Sigma \frac{u\,v\,l}{EA} = -\Sigma_1{}^n \frac{S'\,v\,l}{EA}. \quad (14)$$

From these equations S_r and S_{r+1} may be obtained, and then S from eq. (12).

In a similar manner any number of redundant members may be dealt with, it being always possible to derive as many equations as there are redundant members.

226. The Principle of Least Work Applied to Structures with Redundant Members.—By Art. 213 the expression for deflection, $\Sigma \frac{Sul}{EA}$, may be written $\Sigma \frac{Sl}{EA} \cdot \frac{dS}{dW}$, and is the differential coefficient of the internal work with respect to a force W applied at the joint in question. Taken as explained in Art. 221 this deflection is zero; that is, the first derivative of the internal work, with respect to a force applied at the end of the redundant member, is zero. But if the first derivative of a function is zero, that function is either a maximum or a minimum. In this case it must be a minimum. Hence it may be said that the stresses in a structure with redundant members are so distributed that the internal work of distortion is a minimum. This is the well-known principle of *least work*.

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